Dynamics of a Family of Eighth-Degree Complex Polynomials

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DYNAMICS OF A FAMILY OF EIGHTH-DEGREE COMPLEX POLYNOMIALS

By

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The Department of Mathematics and Computer Science at Dickinson College hereby accepts this senior honors thesis by Simon Feeman, and awards departmental honors in Mathematics.
Dynamics of a Family of Eighth-degree Complex Polynomials

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Simon Feeman

Much work has been done on the dynamics of quadratics and other low degree polynomials. However, less is known about the dynamics of higher degree polynomials as the algebra is much more difficult. In this thesis we study a family of 8th degree polynomials, $P_c : \mathbb{C} \to \mathbb{C}$ where, $P_c(z) = z^2 - cz^8$ and $c \in \mathbb{C} - 0$. In particular we show that the Julia set is either connected or disconnected but not Cantor. Along with this, we investigate interesting properties of the parameter space and its symmetry.
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Dynamical systems is a branch of mathematics used to study how functions, typically from state to state, can change over time. An example is how small changes in input affect the output. In this field, we study the behavior of systems when repeatedly applied with a function. Recurrence refers to studying functions "in action." There are many applications of dynamical systems from population studies to financial markets.

4.1 Dynamical Systems. First we need to define some basic notions and formalisms of dynamical systems. In a class of dynamical systems we will be investigating in this thesis is the class of discrete dynamical systems.

Definition 4.1. Let \( X \) be a topological space. A discrete dynamical system is the pair \((X, f)\), where \( f : X \to X \) is a continuous function. Initially applying \( f \) to a starting value of \( x \) gives its ultimate sequence.

For example, the sequence \( F^n(x) \) is the \( n \)th iterate of \( f \) on \( x \), and \( F^0(x) = x \). The sequence \( F^n(x) \) is called the orbit of \( x \).

Dynamical systems are commonly used to model dynamical systems with discrete time, often to understand the behavior of real-world systems. These systems can be very predictable in some cases and can be very chaotic in others, even when the changes are very small. Examples of dynamical systems include those that describe weather patterns, stock market trends, and the spread of diseases. The study of dynamical systems has led to new insights into the complexity and unpredictability of natural and social phenomena.
1. Introduction

Dynamical systems is a branch of mathematics used to study how functions act from state to state. It analyzes how small changes in input affect the output. In this field we study the behavior of points when repeatedly inputted into a function. Essentially we are studying functions “in motion.” There are many applications of dynamical systems from population studies to financial analysis.

1.1. Dynamical Systems. First we need to define some general terms from dynamical systems. The type of dynamical system we will be investigating in this thesis is a discrete dynamical system.

Definition 1.1. Let $X$ be a topological space. A discrete dynamical system is the process of iterating a function, $f : X \to X$. Repeatedly applying $f$ to a starting value of $x_0$ results in the sequence

$$x_0, f(x_0), f^2(x_0), f^3(x_0) \ldots$$

where $f^n(x_0) = f(f^{n-1}(x_0))$ is the $n$th iterate of $f(x_0)$, and $f^0(x_0) = x_0$. The sequence of iterates of $x_0$ is called the orbit of $x_0$.

In this thesis we focus exclusively on real or complex dynamical systems, so $X = \mathbb{R}$ or $X = \mathbb{C}$ throughout the remainder of this paper. These systems can be very predictable at certain input points and yet very hard to analyze at others, even when the change of input is small. The study of dynamical systems analyzes which inputs follow a predictable pattern and which don’t under iterations. Much of the study of dynamical systems focuses on studying the behavior of particular sets of points when iterated, so we thus look at their orbits.
A lot of interesting behavior in dynamical systems occurs around fixed points and periodic points.

**Definition 1.2.** A point \( p \in X \) is a *fixed point* for \( f \) if \( f(p) = p \). The point \( p \) is a *periodic point of period* \( n \) if \( f^n(p) = p \) and \( n \) is the least such positive integer. We then say that \( p \) lies in an \( n \)-cycle.

We show this in an example using a quadratic polynomial.

**Example 1.3.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = x^2 - 1 \). Then \( p_0 = \frac{1+\sqrt{5}}{2} \) is a fixed point since \( f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2} \). Along with this, the point \( p_1 = -1 \) is a periodic point of period 2 since the orbit of \(-1\) is \(-1, 0, -1, 0, \ldots \). We can see this example below in Figure 1.1.

Now we go on to classify fixed points as follows.

**Definition 1.4.** Let \( f : \mathbb{R} \to \mathbb{R} \). A fixed point \( p \) where \(|f'(p)| > 1\) is called a *repelling fixed point*. When \(|f'(p)| = 1\) the fixed point \( p \) is called *indifferent*. When \(0 < |f'(p)| < 1\), \( p \) is called an *attracting fixed point*, and when \(|f'(p)| = 0\), \( p \) is called a *super attracting fixed point* (SAFP).

Again we illustrate this with our previous quadratic polynomial.

**Example 1.5.** For \( f(x) = x^2 - 1 \), we have \(|f'(\frac{1+\sqrt{5}}{2})| = |1 + \sqrt{5}| > 1\), so \( p_0 = \frac{1+\sqrt{5}}{2} \) is a repelling fixed point. We see this in Figure 1.1.
The following theorems give details about the names used in Definition 1.4. The adjectives “attracting” and “repelling” give information about the orbits of points that are near a fixed point. The proof of Theorem 1.6 is taken from the one in Devaney [4].

**Theorem 1.6.** Suppose $x_0$ is an attracting or super attracting fixed point for $F : \mathbb{R} \to \mathbb{R}$. Then there is an interval $I$ that contains $x_0$ in its interior and in which the following condition is satisfied: if $x \in I$, then $F^n(x) \in I$ for all $n$ and, moreover, $F^n(x) \to x_0$ as $n \to \infty$.

**Proof.** Since $|F'(x_0)| < 1$, then there is a $m > 0$ where $m \in \mathbb{R}$ such that $|F'(x_0)| < m < 1$. We can therefore choose a $q \in \mathbb{R}$ where $q > 0$, so that $|F'(x)| < m$ where $x \in I = [x_0 - q, x_0 + q]$. Now we let $p \in I$. By the Mean Value Theorem we have $|\frac{F(p) - F(x_0)}{p - x_0}| < m$, or, $|F(p) - F(x_0)| < m|p - x_0|$. Since $x_0$ is a fixed point we know that $|F(p) - x_0| < m|p - x_0|$. Therefore since $0 < m < 1$ we know that the distance from $F(p)$ to $x_0$ is smaller than that of $p$ to $x_0$. So $F(p) \in I$, therefore we apply the same argument for $F(p)$ and $F(x_0)$. Then, $|F^2(p) - F(x_0)| = |F^2(p) - F^2(x_0)| < m|F(p) - F(x_0)|$ so we know that $|F^2(p) - F(x_0)| < m^2|p - x_0|$. Since $m < 1$, we know that $m < m^2$, therefore we know that $F^2(p)$ and $x_0$ are even closer together. Thus we may continue to apply this
argument to find that for any $n > 0$, $|F^n(p) - x_0| < m^n|x - x_0|$. So now we have that $m^n \to 0$ as $n \to \infty$. Hence we have that $F^n(p) \to x_0$ as $n \to \infty$. 

A similar proof holds for the following theorem about the repelling fixed points.

**Theorem 1.7.** Suppose $x_0$ is a repelling fixed point for $F : \mathbb{R} \to \mathbb{R}$. Then there is an interval $I$ that contains $x_0$ in its interior and in which the following condition is satisfied: if $x \in I$ and $x \neq x_0$, then there is an integer $n > 0$ such that $F^n(x) \notin I$.

Critical points play an important role in real and complex dynamical systems. Recall that $c$ is a critical point of $f$ if $f'(c) = 0$. We are particularly interested in the case where critical points are periodic.

**Definition 1.8.** A critical point $c_p$ of $f : X \to X$ where, $X = \mathbb{C}$ or $\mathbb{R}$ is in a super attracting $n$-cycle if $c_p = f^n(c_p)$ for the least such positive integer $n$.

For real dynamical systems, bifurcations occur when there is a change in the fixed or periodic point structure when a parameter $c$ passes through some value.

**Definition 1.9.** A one parameter family of functions $F_c : \mathbb{R} \to \mathbb{R}$ undergoes a saddle-node bifurcation at the parameter value $c_0$ if there is an open interval $I$ and $\varepsilon > 0$ such that:

1. For $c_0 - \varepsilon < c < c_0$, $F_c$ has no fixed points in interval $I$.
2. For $c = c_0$, $F_c$ has one fixed point in $I$ and this fixed point is indifferent.
3. For $c_0 < c < c_0 + \varepsilon$, $F_c$ has two fixed points in $I$, one attracting, one repelling.
Similarly we have the following definition to describe what happens when one bifurcation creates a 2-cycle.

**Definition 1.10.** A one parameter family of functions \( F_c : \mathbb{R} \to \mathbb{R} \) undergoes a *period doubling bifurcation* at the parameter value \( c = c_0 \) if there is an open interval \( I \) and \( \varepsilon > 0 \) such that:

1. For each \( c \) in the interval \([c_0 - \varepsilon, c_0 + \varepsilon]\), there is a unique fixed point \( p_c \) for \( F_c \) in \( I \).
2. For \( c_0 - \varepsilon < c \leq c_0 \), \( F_c \) has no cycles of period 2 in \( I \) and \( p_c \) is attracting.
3. For \( c_0 < c < c_0 + \varepsilon \), there is a unique 2 cycle \( q_c^1, q_c^2 \) in \( I \) with \( F_c(q_c^1) = q_c^2 \). This 2-cycle is attracting, while \( p_c \) is repelling.
4. As \( c \to c_0 \), we have \( q_c^1 \to p_c \).

1.2. **Complex Dynamical Systems.** The primary goal of this thesis is to investigate a one parameter family of complex dynamical systems, so we give the necessary background on complex dynamical systems in this section. Much of what we do involving the study of complex functions uses an analysis of the orbits of particular points and more importantly the changes in the Julia set defined below. To analyze these we must first define some mathematical terms. We can study the behavior of each of the fixed points by analyzing the first derivative. A lot of interesting properties in complex dynamical systems result from analyzing the long term behavior of orbits.

**Definition 1.11.** The orbit of \( z \) under \( f : \mathbb{C} \to \mathbb{C} \) is *bounded* if there exists a \( K \in \mathbb{R} \) such that \( |f^n(z)| < K \) for all \( n \geq 0 \). Otherwise, the orbit of \( z \) is *unbounded*.

Next, we present the very important definition of the Julia set.
**Definition 1.12.** The *filled Julia set* of $f$ is the set of all points whose orbits are bounded. The *Julia set* of $f$ is the boundary of the filled Julia set. We denote the Julia set of $f$ as $J(f)$.

In order to talk formally about the structure of the Julia set we must define when a set is connected using the topological definition.

**Definition 1.13.** A set $B$ is *connected* if and only if $B$ cannot be written as the union of two non-empty disjoint open sets.

Next, we must define what it means to be totally disconnected.

**Definition 1.14.** A set of complex numbers is *totally disconnected* if every connected component is a one-point set.

Now we define a set being Cantor using the topological definition.

**Definition 1.15.** A set of complex numbers is a *Cantor set* if it is closed, bounded, totally disconnected, and if every point is an accumulation point of the set.

The following theorem is a classical result in the field of complex dynamical systems, and is due to Julia and Fatou. We do not present a proof here.

**Theorem 1.16.** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial.

1) The Julia set is connected if every critical point has a bounded orbit.

2) The Julia set is Cantor if every critical point has unbounded orbit.
3) The Julia set is disconnected not Cantor if some critical points have bounded orbits and some have unbounded orbits.

The following result for quadratic polynomials is well-known.

**Theorem 1.17.** If \( f \) is a quadratic polynomial, then \( J(f) \) is either Cantor or connected.

However, if we investigate the family of cubic polynomials, we can find examples that fit each of the 3 criteria in Theorem 1.16. The following examples of cubic polynomials appear in Beardon [1].

**Example 1.18.** We take \( f_1(z) = z^3 \), this function has a connected Julia set. The function has a critical point only at 0 (which is a SAFP). So the orbit of the critical point are bounded.

**Example 1.19.** We take \( f_2(z) = (3\sqrt{3}/2)(z^3 + 3z^2 + 2z) \), this function has critical points at \( a = -1 - \frac{1}{\sqrt{3}} \) and \( b = -1 + \frac{1}{\sqrt{3}} \). The orbit of \( a \) goes to infinity, and the orbit of \( b \) goes to 0. The Julia set is disconnected and not Cantor.

**Example 1.20.** Next, we take \( f_3(z) = z^3 - 3z + 1 \). The critical points are 1 and -1, and both iterate to infinity. The Julia set is Cantor.

1.3. **Our Family of Functions.** Much work has been done on the dynamics of quadratics and other low degree polynomials. However, less is known about the dynamics of higher degree polynomials as the algebra is much more difficult. In this thesis we study a family of 8th degree polynomials, \( P_c : \mathbb{C} \to \mathbb{C} \) where, \( P_c(z) = z^2 - cz^8 \) and \( c \in \mathbb{C} \setminus \{0\} \). In the next section we will be analyzing many properties of this family relative to the
value of $c$. We will be studying the behavior of critical points, fixed points, and the relationships between the Julia sets of different values of $c$. 
2. The Function $P_c$

Typically, the first step in investigating a complex dynamical system is to find and classify the fixed points. We begin by attempting to find the fixed points of $P_c(z) = z^2 - cz^8$, but we quickly run into trouble. Note that to solve for the fixed points we must solve the equation, $z = z^2 - cz^8$, and therefore we must solve, $-cz^8 + z^2 - z = 0$. We know that there are 8 complex solutions, one of which is clearly 0, but there is no obvious way to find the other solutions for an arbitrary $c$. Given the high degree of the equation for the fixed points, we must look at other properties of the function to find out more of the behavior of $P_c$.

One nice property of this function is the form of the critical points. To find the critical points, we calculate $P_c'(z) = 2z - 8cz^7$, and thus 0 is a critical point.

**Proposition 2.1.** $P_c$ has a SAFP at $z = 0$ for all $c$.

*Proof.* Since 0 is a critical point and since $P_c(0) = 0$, $z = 0$ is also a fixed point. We then see that 0 is a SAFP by Definition 1.4.

The rest of the critical points are easy to find and yield some very interesting properties.

**Theorem 2.2.** The critical points of $P$ are $p_0 = 0$ and $p_k$ where $p_k = \sqrt[6]{\frac{1}{4c} e^{\frac{\pi i \theta + 2k \pi}{6}}}$ where $\theta = \text{Arg}(c)$ and $k = 1, 2, \ldots, 6$.

*Proof.* Note that $P'(z) = 2z - 8cz^7 = 2z(1 - 4cz^6)$. By Proposition 2.1 we know 0 is a critical point. The rest of the critical points found by solving, $P'(z) = 2z(1 - 4cz^6) = 0$, so $1 = 4cz^6$, or $\frac{1}{4c} = z^6$. Writing $\frac{1}{4c}$ in exponential form, we can then see that $z^6 = \sqrt[6]{\frac{1}{4c} e^{\pi i \theta + 2k \pi}}$ where $\theta = \text{Arg}(\frac{1}{c})$ and $k = 1, 2, \ldots, 6$. Finally $z = \sqrt[6]{\frac{1}{4c} e^{\frac{\pi i \theta + 2k \pi}{6}}}$. \qed
Corollary 2.3. All the nonzero critical points of $P_c$ are $\pi/3$ rotations of each other in the complex plane.

3. Real Values of $c > 0$

Although we are ultimately interested in studying the complex function $P_c$, we can gain insight by beginning with an investigation of the real function $P_c : \mathbb{R} \to \mathbb{R}$ when $c > 0$. In this case, $P_c$ has a unique positive real critical point, which we denote $p_c$ throughout the rest of this section. For example, when $c = \frac{1}{4}$ we then see that $P_{\frac{1}{4}}(z) = z^2 - \frac{1}{4}z^8$ and $P'_{\frac{1}{4}}(z) = 2z - 2z^7 = 2z(1 - z^6)$. So it is clear that $p_{\frac{1}{4}} = 1$. We begin by proving some basic facts about where $P_c$ is increasing and decreasing.

Lemma 3.1. When $c > 0$, $P_c$ is increasing when $x \in [0, p_c]$ and is decreasing when $x \in [p_c, \infty)$.

Proof. Since $P_c'$ has roots at $x = 0$ and $x = p_c$ we pick points in the intervals $[0, p_c]$ and $[p_c, \infty)$ and test the sign of $P_c'$ at those points. Let $c > 0$, thus $P_c'(x) = 2x(1 - 4cx^6)$ with a positive real critical point, $p_c = \frac{\sqrt[3]{4}}{4c}$. Take $\frac{p_c}{2} \in [0, p_c]$, and notice that,

$$P_c'(\frac{p_c}{2}) = P_c'\left(\frac{1}{2} \sqrt[3]{\frac{1}{4c}}\right) = \frac{\sqrt[3]{4}}{4c} (1 - 4c(\frac{1}{4c} \cdot \frac{1}{64})) = \frac{\sqrt[3]{4}}{4c} (1 - \frac{1}{64}) > 0.$$

Thus, $P_c$ is increasing when $x \in [0, p_c]$. Now take $2p_c \in [p_c, \infty)$, notice that $P_c'(2 \sqrt[3]{\frac{1}{4c}}) = 4 \sqrt[3]{\frac{4}{4c}} (1 - 4c(64 \frac{1}{4c})) = 4 \sqrt[3]{\frac{4}{4c}} (-63) < 0$. Thus, $P_c$ is decreasing when $x \in [p_c, \infty)$. $\square$

3.1. Parameters with no positive fixed point in $\mathbb{R}$. We are interested in finding values of $c$ for which $P_c$ has no positive real fixed points. We begin by investigating the case where $c = \frac{1}{4}$.
Lemma 3.2. There is no positive real fixed point for $P_{\frac{1}{4}}$.

Proof. Recall that $p_{\frac{1}{4}} = 1$. Let $x \in [0, p_{\frac{1}{4}}]$. Since $x^2 \leq x$, $P_{\frac{1}{4}}(x) = x^2 - \frac{1}{4}x^8 < x^2 \leq x$. So $P_{\frac{1}{4}}(x) < x$. By Lemma 3.1 $P_{\frac{1}{4}}(x)$ is decreasing on $[p_{\frac{1}{4}}, \infty)$, so for all $x \in (0, \infty)$, $P(x) < x$. Therefore, there is no positive real fixed point.

We show the graph of $P_{\frac{1}{4}}(x)$ in blue and the line $y = x$ in red in Figure 3.2.

![Figure 3.2. The graph of $P_{\frac{1}{4}}(x) = x^2 - \frac{1}{4}x^8$.](image)

We can generalize this result for a large set of values for $c$.

Theorem 3.3. When $c \geq \frac{1}{4}$ there is no positive real fixed point for $P_c$.

Proof. Let $c \geq \frac{1}{4}$. Thus $P_c(x) = x^2 - cx^8 \leq x^2 - \frac{1}{4}x^8$. So, $P_c(x) \leq P_{\frac{1}{4}}(x) < x$. So for all $x \in (0, \infty)$, $P_c(x) < x$. Thus, there is no positive real fixed point.

Next, we have some results on the orbit of any point in $[0, P_c]$.

Corollary 3.4. When $c \geq \frac{1}{4}$ and $x \in [0, p_c]$, the orbit $\{P_c^n(x)\}$ is decreasing and bounded.

Proof. We know that when $x \in [0, p_c]$, $P_c(x) < x$. But $P_c(x) > 0$ by the definition of our function, so $P_c(x) \in [0, p_c]$, which implies $P_c^2(x) = P_c(P_c(x)) < P_c(x)$. Using induction, we know that $P_c^n(x)$ is decreasing. Similarly, since $P_c(x) \in [0, p_c]$ for all $n > 0$, $P_c^n(x)$ is bounded below by 0.

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In fact, we can apply the above theorem and corollary to prove that if \( c \geq \frac{1}{4} \), then all points in \([0, p_c]\) have orbits that approach 0.

**Theorem 3.5.** For all \( c \geq \frac{1}{4} \), and \( x \in [0, p_c] \), \( P^n_c(x) \to 0 \).

**Proof.** We know by Corollary 3.4 that \( \{P^n_c(x)\} \) is decreasing and bounded, thus it converges. Now we suppose that for \( x \in [0, p_c] \), \( P^n_c(x) \to q \) where \( q \geq 0 \). Therefore by definition, we have the sequence \( x, P_c(x), P^2_c(x), \ldots \to q \). Applying \( P \) to each point in the sequence we then obtain, \( P_c(x), P^2_c(x), P^3_c(x) \ldots \to P_c(q) \). This implies that \( P_c(q) = q \), so \( q \) is a fixed point. By Theorem 3.3, \( q = 0 \). Therefore, when \( x \in [0, p_c] \), \( P^n_c(x) \to 0 \).

\( \Box \)

For instance, as we see in Figure 3.3 \( P_1(x) = x^2 - x^8 \) has no positive fixed points. In this figure the line \( y = x \) appears in red and the graph of \( P_1(x) \) appears in blue. The intersection of these two graphs are the fixed points.

---

**Figure 3.3.** The graph of \( P_1(x) = x^2 - x^8 \).

### 3.2. Parameters with a Saddle-Node Bifurcation.

However, we know that at some point, as \( c \) decreases, there is a real fixed point (the line \( y = x \) intersects \( P_c \) at some positive \( x \)). As seen earlier, we cannot solve for this point algebraically because the degree of the equation is too large. Using Mathematica we estimate that it occurs near
\( c = .06 \), as seen in Figure 3.4. In this case, the fixed point is neutral since the derivative is 1 at the tangent point.

![Figure 3.4](image)

**Figure 3.4.** The graph of \( P_{.06}(x) = x^2 - .06x^8 \).

As \( c \) continues to decrease, \( P_c \) will have two positive real fixed points. In Figure 3.5 we show the function \( P_{.05}(x) = x^2 - .05x^8 \). In this case, the smaller of the two real fixed points is repelling, and the larger of the two is attracting.

![Figure 3.5](image)

**Figure 3.5.** The graph of \( P_{.05}(x) = x^2 - .05x^8 \).

Figure 3.6, is a zoom in on the domain where both positive real fixed points occur.
Therefore as \( c \) decreases through approximately .06, we go from having no positive fixed points to one neutral fixed point, to two positive real fixed points, one attracting, the other repelling. Thus by Definition 1.9 we know that we have a saddle-node bifurcation at approximately \( c = .06 \).

### 3.3. Parameters with a nonzero super attracting fixed point.

In fact, there are values of \( c > 0 \) for which \( P_c \) has a nonzero SAFP, and we can solve for them explicitly.

**Theorem 3.6.** When \( c = \frac{1}{4}(\frac{3}{4})^6 \) we have a super attracting fixed point at \( z = 4/3 \).

*Proof.* Let \( c = \frac{1}{4}(\frac{3}{4})^6 \). The positive real critical point of \( P_c \) is given by \( p_c = \sqrt[6]{\frac{1}{4c}} \). Thus,

\[
p_c = \sqrt[6]{\frac{1}{4(\frac{1}{4}(\frac{3}{4})^6)}} = \frac{4}{3}.
\]

Notice then that, \( P_c(4/3) = (4/3)^2 - 1/4(3/4)^6(4/3)^8 = 16/9 - 4/9 = 4/3 \). Therefore \( z = 4/3 \) is a SAFP.
Using *Mathematica*, the graph of $P_c$ in the real plane when $c = (\frac{1}{4})(\frac{3}{4})^6$ is shown in Figure 3.7. We also show an enlarged image on the domain of the function near the super attracting fixed point in Figure 3.8.

![Graph of $P_c$](image)

**Figure 3.7.** The graph of $P_{\left(\frac{1}{4}\right)^6}(x) = x^2 - (\frac{1}{4})(\frac{3}{4})^6x^8$.

![Enlarged image](image)

**Figure 3.8.** $P_{\left(\frac{1}{4}\right)^6}(x) = x^2 - (\frac{1}{4})(\frac{3}{4})^6x^8$ at the SAFP.

We can see that the function $P_{\left(\frac{1}{4}\right)^6}$ intersects $y = x$ at its maximum, and therefore the critical point is a fixed point.

### 3.4. Super attracting two-cycle

Similarly we can look at the real positive critical point and solve for a super attracting 2-cycle.

**Lemma 3.7.** For $c = \frac{81(7+\sqrt{13})}{32768}$, $P_c$ has a super attracting 2-cycle at the critical point $p_c = (4(2/(7 + \sqrt{13}))^{1/6})/(3^{2/3})$. 
Proof. Let \( c = \frac{81(7+\sqrt{13})}{32768} \). Therefore our critical point is \( p_c = \frac{4(2/(7 + \sqrt{13}))^{1/6}}{(3^{2/3})} \).

Now through extensive calculation we see that

\[
P_c^2 \left( \frac{2}{(7+\sqrt{13})}^{1/6} \right) = \frac{4(2/(7 + \sqrt{13}))^{1/6}}{(3^{2/3})}.
\]

So \( P_c \) has a super attracting 2-cycle at the critical point \( p_c = \frac{4(2/(7 + \sqrt{13}))^{1/6}}{(3^{2/3})} \).

In Figure 3.9 we show the graph of \( P_{\frac{81(7+\sqrt{13})}{32768}} \) in blue. It is difficult to see the two cycle in the figure, so we show the graph of \( P_{\frac{81(7+\sqrt{13})}{32768}}^2 (x) \) in green. Also, we show the two graphs on a smaller part of the domain in Figure 3.10. The fixed points of \( P_{\frac{81(7+\sqrt{13})}{32768}}^2 (x) \) that are not fixed points of \( P_{\frac{81(7+\sqrt{13})}{32768}} \) correspond to the super attracting 2-cycle for \( P_{\frac{81(7+\sqrt{13})}{32768}} \).

![Figure 3.9](image-url)
If we attempt to solve for a super attracting 3-cycle we must solve a 6th degree polynomial equation which will not reward an exact answer.

3.5. Schwarzian derivative. The Schwarzian derivative is a tool that can give us information about the number of attracting cycles a polynomial can have. The definition of the Schwarzian derivative is as follows. The \textit{Schwarzian derivative} of any polynomial $P$ is given by

$$S_P(x) = \frac{P''(x)}{P'(x)} - \frac{3}{2} \left( \frac{P''(x)}{P'(x)} \right)^2.$$ 

Functions with negative Schwarzian derivatives satisfy some very nice properties. In particular, attracting periodic points are associated with critical points.

\textbf{Definition 3.8.} The \textit{basin of attraction} of the attracting periodic point $x_0$ is the set of all $x \in \mathbb{R}$ whose orbits tend to $x_0$. The \textit{immediate basin of attraction} of $x_0$ is the largest interval containing $x_0$ that lies in the basin of attraction.

The following theorem was proved by Singer [6] and a simpler proof appears in Devaney [4].

\textbf{Theorem 3.9.} When $S_P < 0$ for all $x$, given an attracting periodic point $x_0$ of $P$, then either the immediate basin of attraction of $x_0$ extends to $\infty$ or $-\infty$ or else there is a critical point of $P$ whose orbit is attracted to the orbit of $x_0$. 

\textbf{Figure 3.10.} $P_{\frac{31(7+\sqrt{13})}{32768}}(x) = x^2 - \left( \frac{31(7+\sqrt{13})}{32768} \right)^6 x^8$ and $P_{\frac{31(7+\sqrt{13})}{32768}}^2(x)$ at the super attracting two cycle.
Functions with non-negative Schwarzian derivatives can have attracting periodic points that are not associated with critical points. We will proceed with an example from Singer [6].

**Example 3.10.** Consider the function \( F(x) = 7.86x - 23.31x^2 + 28.75x^3 - 13.30x^4 \) defined on \([0, 1]\). The Schwarzian derivative and we get \( S_f(x) = -\frac{1.5(-46.62 + 172.5x - 159.6x^2)^2}{(7.86 - 46.62x + 86.25x^2 - 53.2x^3)^2} + \frac{172.5 - 319.2x}{7.86 - 46.62x + 86.25x^2 - 53.2x^3} \), notice this is not always negative. We can see from the figure below that there are attracting fixed points at \( x = 0 \) and \( x = 0.726399 \) but only one critical point, at \( x = 0.32398 \). The critical point at \( x = 0.32398 \) iterates to 0. So this function has an attracting fixed point at \( x = 0.726399 \) that does not attract a critical point.

![Figure 3.11. The graph of \( F(x) = 7.86x - 23.31x^2 + 28.75x^3 - 13.30x^4 \).](image)

Next, we investigate our family of functions using Singer’s approach. Our goal is to find a bound on the number of attracting fixed points of \( P_c \) using Singer’s Theorem. Through some extensive calculations we get the the Schwarzian derivative of our function \( P_c(x) = x^2 - cx^8 \).

**Theorem 3.11.** The Schwarzian derivative of \( P_c \) is given by \( S_{P_c}(x) = -\frac{\frac{3}{2} + 84cx^6 + 504c^2x^{12}}{x^2(1 - 4cx^2)^2} \).

Further, when \( x \neq 0 \) and \( x \neq p_c \), \( S_{P_c}(x) < 0 \).
Proof. We know that $P_c'(x) = 2x - 8x^7$, $P_c''(x) = 2 - 56x^6$, and $P_c'''(x) = -336x^5$.

Therefore, when we plug this into the definition we get $S_{P_c}(x) = \frac{-336x^3}{2x-8x^7} - \frac{3}{2}(\frac{2-56x^6}{2x-8x^7})^2 = -\left(\frac{\frac{3+84cx^6+504c^2x^{12}}{x^2(1-4cx^6)^2}}{x^2(1-4cx^6)^2}\right)$. Moreover, since every $x$ term is raised to a even power and $c > 0$, we then know that $S_{P_c}(x) < 0$. 

Since our function is a polynomial, we also have the following lemma.

**Lemma 3.12.** The basin of attraction of an attracting periodic point for a polynomial cannot extend to $\infty$.

**Proof.** Since

$$\lim_{x \to \pm \infty} P_c(x) = \pm \infty,$$

an attracting basin cannot extend to $\infty$. 

Because of Lemma 3.12 we are then left with the following theorem.

**Theorem 3.13.** For real $c > 0$, $P_c(z) = z^2 - cz^8$ has at most 2 real attracting periodic points.

**Proof.** We know that the basin of attraction cannot extend to infinity for our function $P_c$. Therefore by Theorem 3.9 any attracting periodic cycle must attract a critical point. Although $P_c$ has 3 real critical points, $P_c(p_c) = P_c(-p_c)$, so it only has two distinct periodic orbits. Since $P_c$ has only two real critical orbits, there are at most 2 real attracting periodic points.
4. Complex Values of $c$

We now have the capabilities to analyze properties of $P_c$ when $c \in \mathbb{C}$. We can see that some interesting properties occur based on the symmetry of our function.

4.1. Symmetries. Beyond the rotational property of the critical points described in Corollary 2.3 we notice a symmetry in the orbits of any $z$ under $P_c$.

**Theorem 4.1.** For any $z \in \mathbb{C}$ and $n > 0$,

$$P_c^n(e^{(\pi i)/3}z) = \begin{cases} 
  e^{(4\pi i)/3}P_c^n(z) & \text{if } n \text{ is even} \\
  e^{(2\pi i)/3}P_c^n(z) & \text{if } n \text{ is odd}.
\end{cases}$$

**Proof.** Notice that $P_c(z) = z^2 - cz^8$, so $P_c^2(z) = (z^2 - cz^8)^2 - c(z^2 - cz^8)^8$. Therefore,

$$P_c(e^{\frac{\pi i\theta}{3}}z) = e^{\frac{2\pi i\theta}{3}}z^2 - e^{\frac{8\pi i\theta}{3}}cz^8 = e^{\frac{2\pi i\theta}{3}}(z^2 - cz^8) = e^{\frac{2\pi i\theta}{3}}P_c(z).$$

We then see that

$$P_c^2(e^{\frac{\pi i\theta}{3}}z) = (e^{\frac{2\pi i\theta}{3}}(z^2 - cz^8))^2 - c(e^{\frac{2\pi i\theta}{3}}(z^2 - cz^8))^8 = e^{\frac{4\pi i\theta}{3}}((z^2 - cz^8)^2 - c(z^2 - cz^8)^8) = e^{\frac{4\pi i\theta}{3}}P_c^2(z).$$

Since $P_c^{n+1} = P_c(P_c^n(z))$, this pattern will continue by induction. Therefore,

$$P_c^n(e^{(\pi i)/3}z) = \begin{cases} 
  e^{(4\pi i)/3}P_c^n(z) & \text{if } n \text{ is even} \\
  e^{(2\pi i)/3}P_c^n(z) & \text{if } n \text{ is odd}.
\end{cases}$$

Given this relationship of the orbits, we can see that the Julia sets will exhibit a rotational symmetry.

**Theorem 4.2.** $J(P_c) = e^{\frac{\pi i}{3}}J(P_c)$
Proof. We know that there exists some bound $K$ for all $z$ in the filled Julia set, $|P_c^n(z)| < K$ for all $n \geq 0$. We know by Theorem 4.1

$$P_c^n(e^{(\pi i)/3}z) = \begin{cases} 
  e^{(4\pi i)/3}P_c^n(z) & \text{if } n \text{ is even} \\
  e^{(2\pi i)/3}P_c^n(z) & \text{if } n \text{ is odd.}
\end{cases}$$

Thus, since $P_c^n(z)$ is bounded by $K$, $|P_c^n(e^{(\pi i)/3}z)| = |P_c^n(z)| \leq K$. Hence, if $z$ is in the filled Julia set, $e^{(\pi i)/3}z$ is in the filled Julia set. Therefore, the filled Julia set of $P_c$ is equal to the filled Julia set of $e^{(\pi i)/3}P_c$, and we see that, $e^{(\pi i)/3}J(P_c) = J(P_c)$. \qed

4.2. Structure of the Julia set. Next we examine some topological properties of the Julia set. Although Theorem 1.17 implies that the Julia set of a polynomial can be connected, Cantor, or disconnected but not Cantor, we don’t actually see all types of behavior in the family $P_c$.

**Theorem 4.3.** For all $c \in \mathbb{C}$, $J(P_c)$ is not Cantor.

**Proof.** We know by Corollary 2.1 that $z = 0$ is always a SAFP, thus $z = 0$ is a critical point whose orbit is always bounded, so $J(P_c)$ is not Cantor by Theorem 1.16. \qed

We can then restrict our focus by proving that certain behaviors occur for particular values of $c$. We begin with a result for real $c$.

**Theorem 4.4.** For real $c$-values with $c > \frac{1}{4}$, $J(P_c)$ is connected.

**Proof.** We know by Theorem 3.5 that when $c > \frac{1}{4}$ the positive real critical point iterates to 0, thus its orbit is bounded. Moreover, we know by Theorem 4.1 that if one critical point has bounded orbit, then all of them do, so $J(P_c)$ is connected. \qed
In fact, we can extend this result to a larger region of the complex plane.

**Theorem 4.5.** For any \( c \) with \( |c| > \frac{1}{4} \), \( J(P_c) \) is connected.

**Proof.** Let \( |c| > \frac{1}{4} \). Then \( |p_c| = \frac{6}{|4c|} \). But since \( |c| > \frac{1}{4} \), \( \frac{1}{4c} < 1 \), so \( |p_c| < 1 \). Now take any \( z \) where \( |z| < |p_c| \). We know that \( |P_c(z)| = |z^2 - cz^8| \), but since \( |z| < 1 \) we know that \( |z^2| < 1 \) so it follows that \( |P_c(z)| < |z| \). Therefore, we know that \( \{|P_c^n(z)|\} \) is a decreasing sequence of real numbers that is bounded below. Then, by an argument similar to the proof of Theorem 3.5, we know that \( |P_c^n(z)| \to 0 \), which means that \( P_c^n(z) \to 0 \). \( \square \)

In general, we can use the following Escape Criterion to determine whether a point has an unbounded orbit.

**Theorem 4.6.** *(Escape Criterion)* If \( |z| > \max\{1, \sqrt[6]{\frac{3}{|c|}} \} \), then \( |P_c^n(z)| \to \infty \).

**Proof.** Let \( |z| > \max\{1, \sqrt[6]{\frac{3}{|c|}} \} \). We see that \( |1 - cz^6| = |cz^6 - 1| \) and since \( |z| > \sqrt[6]{\frac{3}{|c|}} \), we know that \( |cz^6 - 1| > 1 \). Now,

\[
|P_c(z)| = |z^2 - cz^8| \\
= |z^2||1 - cz^6| \\
> |z||cz^6 - 1|, \quad \text{since } |z^2| > |z| \text{ because } |z| > 1.
\]

We know that since \( |cz^6 - 1| > 1 \) by assumption, then we can find a \( \lambda \) such that \( |cz^6 - 1| > \lambda > 1 \). Thus \( |P_c(z)| > |z||cz^6 - 1| > \lambda |z| \). We repeat this process \( n \) times and we see that \( |P_c^n(z)| > \lambda^n |z| \). Since \( \lambda > 1 \),

\[
\lim_{n \to \infty} |P_c^n(z)| > \lim_{n \to \infty} \lambda^n |z| \to \infty.
\]
Thus the orbit of $z$ is unbounded.

To draw our pictures of Julia sets we test values of $z$, iterate them, and apply the Escape Algorithm. If $z$ goes to zero after around 10 iterates, we color the pixel $z$ dark purple. We color the pixel $z$ other shades of purple when $z$ iterates to another fixed point or cycle other than 0. The tan pixels are all points that iterate to infinity using the Escape Criterion.

4.3. **Examples With Interesting Properties.** Next we look at Julia sets from cases when the Julia set is connected and when its disconnected.

**Example 4.7.** Let $P_{.005}(z) = z^2 - .005z^8$. We know from Section 3.2 that since $.005 < .06$ we have 2 positive real fixed points. Similarly, since $c > 0$ we know that the Schwarzian derivative is negative by Theorem 3.11. Therefore by Lemma 3.12 and Lemma 3.13 the basin of attraction does not extend to infinity and there are at most 2 attracting periodic points.
We know by the escape criterion that since $|P_{.005}(\sqrt[3]{\frac{1}{4c}})| = | - 9.34997 | > \frac{6}{\sqrt{c}}$, all of our nonzero critical points escape to infinity. Therefore the Julia set when $c = .005$ is disconnected not Cantor by Theorem 1.16. We show the filled Julia set of $P_{.005}$ in Figure 4.12. The Julia set is the boundary between the dark purple and tan regions. All points colored purple iterate to the origin, and all points colored tan iterate to $\infty$.

Next we look at the example of when $c = 1/4(3/4)^6$.

**Example 4.8.** Let $c = \frac{1}{4}(\frac{3}{4})^6$. Since $c < .06$, there are 2 nonzero real fixed points by Section 3.2. We know by Theorem 3.6 that $P_{\frac{1}{4}(\frac{3}{4})^6}$ has a super attracting fixed point at $z = 4/3$. Similar to above, the same properties resulting from the Schwarzian derivative hold, namely that $z = 0$ and $z = 4/3$ are the only real attracting fixed points.
We know by Theorem 4.1 that since one critical point has a bounded orbit, all do.

We then notice that given our rotational symmetry we have the following theorem.

**Theorem 4.9.** When \( c = \frac{1}{4}(\frac{3}{4})^6 \), \( P_c \) has 2 superattracting fixed points, 0 and \( p_c \), one eventual fixed point at \(-p_c\), one super attracting two cycle at \( \{e^{\frac{2\pi}{3}}p_c, e^{\frac{4\pi}{3}}p_c\} \), and two eventually periodic critical points at \( e^{\frac{\pi}{3}}p_c \) and \( e^{\frac{5\pi}{3}}p_c \).

**Proof.** Take the real critical point \( p_c \). We know that \( p_c \) is a SAFP and that 0 is a SAFP. We know by Theorem 4.1 that \( P_c^n(e^{\frac{\pi}{3}}p_c) \) falls into the cycle \( \{e^{\frac{2\pi}{3}}p_c, e^{\frac{4\pi}{3}}p_c\} \), which are critical points. So this is a super attracting 2 cycle. After one iteration, \( e^{\frac{\pi}{3}}p_c \) lands in this cycle. Therefore, we know the same holds for \( e^{\frac{5\pi}{3}}p_c \). Similarly, since our function is even, \( P_c(-p_c) = p_c \). Therefore, \(-p_c\) is eventually fixed.

\( \square \)
This is evident in Figure 4.15, where each critical point is in a purple region. All dark purple points have orbits that approach 0. All medium purple points have orbits that approach the SAFP at $z = 4/3$. All light purple points have orbits that approach the two cycle at \( \{e^{\frac{2\pi}{3}}, e^{\frac{4\pi}{3}}\} \). Therefore, we know that the Julia set is connected in this case.

Next we analyze the special example of when $c = \frac{81(7 + \sqrt{13})}{32768}$.

**Example 4.10.** By Lemma 3.7 we know then that $P_{\frac{81(7 + \sqrt{13})}{32768}}$ has a super attracting 2-cycle. Similarly, since $c < .06$ there are two real fixed points. Again, since $c$ is positive we know the same properties of the Schwarzian derivative above hold. Along with this we know that there is a SAFP at the origin. The two positive real critical points go to a super attracting two-cycle at approximately \( \{1.588, 1.456\} \). The other two nonzero critical points approach a super attracting two-cycle at \( \{-0.794 + 1.376i, -0.728 + 1.26i\} \). We know these properties by Theorem 4.1.

![Figure 4.14](image.png)

**Figure 4.14.** The filled Julia set when $c = \frac{81(7 + \sqrt{13})}{32768}$.

Since we know that the real critical point has a bounded orbit, this implies that the Julia set is connected by Theorem 1.16. All points colored the darkest purple have orbits that
approach 0. The other three shades of purple indicate points that head to some 2-cycle. The tan points once again go to infinity.

Finally, we proceed with an example where $c = 1$.

**Example 4.11.** We know that $c > 1/4$, so there are no positive real fixed points by Theorem 3.3. Along with this, $c$ is still positive, so we know the Schwarzian derivative is negative. So there are no nonzero attracting periodic points.

![Figure 4.15](image)

**Figure 4.15.** The filled Julia set when $c = 1$.

We can see that since $c > 1/4$ our Julia set is connected by Theorem 4.5.

Together these examples illustrate the many different types of Julia sets we can have when $c$ changes.
5. **PARAMETER SPACE**

Next, we turn to an investigation of the parameter space of $P_c$. The parameter space shows the properties of the Julia set for different values of $c$. In particular, our parameter space images show $c$ values for which the nonzero critical points of $P_c$ iterate to the origin, iterate to a nonzero cycle, or when they iterate to infinity. The first two cases result in connected Julia sets by Theorem 1.16. The last case indicates when the Julia set is disconnected, not Cantor. In this case, the parameter space shows the connectivity of the family $P_c$.

The following pictures are drawn by iterating the critical points. If the orbit of the nonzero critical points go to 0, we color the pixel $c$ light purple. If the orbit of the nonzero critical points head to some nonzero attracting cycle, we color the pixel $c$ dark purple. If the orbit of the nonzero critical points escape to infinity we color the pixel $c$ tan.

In Figure 5.16 we have the picture of the whole parameter space. Notice that outside of these bounds all pixels are light purple because there is no change in behavior, the Julia set is always connected by Theorem 4.5. So we focus on this area.
We zoom in on two areas where the critical points are iterating to a nonzero cycle, as seen in Examples 4.8 and 4.10.

Recall from Example 4.8 that $P_c$ has a SAFP when $c = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 \approx 0.0444946289$, which is at the center of this picture, in the large purple area in Figure 5.17. In addition, in
Example 4.10, we saw that $P_c$ has a super attracting two-cycle when $c = \frac{81(7+\sqrt{13})}{32768} \approx 0.0262161149082209$, which is located in the “bud” off the main purple area in Figure 5.17. Figure 5.18 shows the parameters where the critical points of $P_c$ are going to super attracting six-cycles. As stated earlier the computation to find a 6-cycle becomes very difficult so this was computed using Mathematica.

![Figure 5.18](image_url)

**Figure 5.18.** The parameters where there is some periodic cycle.

Due to the large degree of $P_c$, we were unable to explicitly solve for the super attracting 3-cycle.

Looking at the images above we notice a symmetry with respect to the real axis. We investigate a relation between the Julia sets of $P_c$ and $P_\bar{c}$.

**Theorem 5.1.** If $z \in J(P_c)$ then $\bar{z} \in J(P_{\bar{c}})$. That is, $J(P_c) = \overline{J(P_{\bar{c}})}$

**Proof.** We know that $P_{\bar{c}}(\bar{z}) = \bar{z}^2 - \bar{c}(\bar{z}^8) = \overline{\bar{z}^2 - \bar{c}(\bar{z}^8)} = \overline{\bar{z}^2 - cz^8} = \overline{P_c(z)}$. Therefore, $P^n_{\bar{c}}(\bar{z}) = \overline{P^n_c(z)}$. Thus, if $|P^n_c(z)| \leq M$ for all $n \geq 0$ we notice that $|P^n_{\bar{c}}(\bar{z})| = |\overline{P^n_c(z)}| = ...$
$|P^n_c(z)| \leq M$ for all $n \geq 0$. So if $z$ is in the filled Julia set of $P_c$ then $\bar{z}$ is in the filled Julia set of $P_{\bar{c}}$. Therefore, $J(P_c) = \overline{J(P_{\bar{c}})}$.

**Corollary 5.2.** $J(P_c)$ is connected if, and only if, $J(P_{\bar{c}})$ is connected.

Thus, we know that the parameter space has symmetry of shape. Next, we prove that the parameter space has symmetry of color. To do this we examine the relationship between cycles and their conjugates. In the next few proofs we often use the property that $P^n_c(z) = P^n_{\bar{c}}(\bar{z})$.

**Theorem 5.3.** If $\{q_1, q_2, \ldots, q_n\}$ is an n-cycle for $P_c$, then $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$ is an n-cycle for $P_{\bar{c}}$. Furthermore, $\{q_1, q_2, \ldots, q_n\}$ has the same classification as $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$.

**Proof.** Let $\{q_1, q_2, \ldots, q_n\}$ be an n-cycle for $P_c$. We know that $P^n_c(q_i) = q_i$ therefore, $P^n_{\bar{c}}(q_i) = \bar{q}_i$. By our statement earlier we then know that $P^n_{\bar{c}}(q_i) = P^n_{\bar{c}}(\bar{q}_i) = \bar{q}_i$. Therefore, $\{\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\}$ is an n-cycle for $P_{\bar{c}}$. Moreover, we know that $(P^n_{\bar{c}})'(\bar{q}_i) = P'_c(\bar{q}_1)P'_c(\bar{q}_2)\ldots P'_c(\bar{q}_n)$ by the chain rule. We know also that $|P'_c(\bar{q}_1)P'_c(\bar{q}_2)\ldots P'_c(\bar{q}_n)| = |P_c(q_1)P_c(q_2)\ldots P_c(q_n)| = |(P^n_c)'(q_i)|$.

Therefore, every n-cycle has the same classification.

Now we investigate the critical points of $P_{\bar{c}}$.

**Lemma 5.4.** The critical points of $P_{\bar{c}}$ are $\overline{p_c}$.

**Proof.** Take $P_{\bar{c}} = z^2 - cz^8$. We know that $p_{\bar{c}} = \sqrt[4]{\frac{1}{16}e^{i\theta + 2k\pi}}$, which is obviously equal to $\overline{p_c}$.

Next we look at the relationships between the orbits of the critical points.
Lemma 5.5. $P^m_c(\overline{p}_c) = P^m_c(p_c)$.

This lemma follows from the statement made about the conjugate of polynomials before this series of proofs. Now we can say something about the symmetry of the coloring of our parameter space.

Theorem 5.6. The parameter space is symmetric with respect to the real axis.

Proof. By Theorem 5.3 we know that the cycles of $\overline{c}$ and $c$ are symmetric over the real axis. By Lemmas 5.4 and 5.5 we know that the critical points of $P_c$ and $P_{\overline{c}}$ have the same behavior so the parameter space is symmetric over the real axis.

Overall, the parameter space really brings a lot together because it shows where each value of $c$ that yields interesting properties is and how their critical points behave. Again, we restrict our attention here because we know by Theorem 4.5 that outside of this area all $c$ values give rise to functions $P_c$ that have a connected Julia set.

6. Future Directions

We have proved a lot about this particular family of functions so far, but with more time we could go even further. One particular direction we could investigate would be changing the degree of our function. For instance instead of $P_c(z) = z^2 - cz^8$ we could analyze $Q_c(z) = z^3 - cz^9$. We can see that $Q'_c(z) = 3z^2(1 + 3cz^6)$. Therefore the critical points will satisfy a symmetry property similar to those of $P_c$. In general we can keep increasing the degree and analyze the family $Q_{c,d}(z) = z^{2+d} + cz^{8+d}$. Clearly, the critical points of the functions satisfy the same rotational property, and the functions still have a SAFP at 0. On the other hand, it isn’t immediately clear how the orbits of the critical
points, in addition to other properties investigated in this thesis, will change. We could further generalize this by examining families of the form $R_{c,d,k}(z) = z^{2+d} + cz^{k+d}$. This would change the number of nonzero critical points, and might lead to other interesting results.
REFERENCES


