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BRACKETING UTILITY

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Abstract

Preferences are the primitive notion in consumer theory. Preferences exist. By contrast, utility is an artificial construct used by economists to represent the underlying primitive notion of preferences. Utility functions are a convenient numerical construct that represents preferences. Utility functions are computationally efficient because they reduce the complexity of solving the consumer's constrained optimization problem. If we have a utility function then the underlying preference map is readily determined by mapping level sets of the utility function. A more interesting question is whether we can always find a utility function that represents a given a set of preferences? If preferences are monotonic and can be represented by indifference curves then it is easy to show it can be represented by a utility function using the 45^o line. Even without indifference map of preferences, for example, if one only knew discrete indifferent bundles, then one can use monotonicity and convexity to bracket utility using a strategy similar to that used in creating a utility function using the 45^o line. In the process, students gain a deeper understanding of monotonicity and convexity. This article focuses attention on monotonicity and convexity and provides a methodology for linking utility to preferences in the event that you do not have a geometric representation of the preference map via indifference curves.

Keywords: Monotonicity, Convexity, Revealed Preference, Existence of utility function, Preferences

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INTRODUCTION

Preferences are the primitive notion in consumer theory. For preferences to be well-defined, they must be subject to a few simple properties including completeness, reflexivity, transitivity, and continuity. Two final properties are required to make them well-behaved, monotonicity and convexity. This article focuses attention on monotonicity and convexity and provides a methodology for linking utility to preferences in the event that you do not have a geometric representation of the preference map via indifference curves. This methodology is similar in flavor to the recoverability analysis which uses revealed preference theory to provide bounds or trap an indifference curve.(Pindyck, R. S., & Rubinfeld, D. L., 2013; Varian, H. R., 1992; Varian, H. R., 2003)

Preferences are, at their most basic level, simply pair-wise comparisons of bundles of goods: I prefer A to B (A≻B); I prefer B to A(B≻A); or, I am indifferent between A and B(A~B). This makes preferences rather cumbersome to work with in practice. Economists have developed geometric and algebraic methods to make the analysis of consumer choice less cumbersome. Indifference curves are a geometric construct and utility functions are an algebraic construct that allow you to represent preferences.

Beginning and intermediate microeconomics students have an easier time with the notion of preferences than utility due to the artificiality of utility. Most intermediate texts move quickly from preferences and indifference curves to the numerical representation of preferences as utility. Students learn that if you have a utility function that represents a set of preferences than any strictly monotonic transformation of that utility function represents the same set of preferences. Put another way, utility is an ordinal, not a cardinal, concept. Indifference curves are readily described from the level sets of a utility function and these level sets are invariant with regard to monotonic transformation.

The reverse side of the coin presents an interesting question. Given a set of preferences, can we always find a utility function that represents these preferences? That is the question examined in this paper. The rest of the paper is written at the level of an intermediate microeconomics text. The goal is to provide students with the ability to more deeply understand convexity and monotonicity. Also included is a URL which provides an Excel file that can be provided to students for their use as well as suggested homework questions based on this analysis. This file includes much of the textual material below so that an instructor could simply distribute the Excel file as an external reading assignment.

The standard analysis of utility functionsfocuses on the algebraic representation of preferences. This focus, of course, begs the question of where these functions came from in the first place. Since utility is ordinal, no single utility function represents a given set of preferences. Notwithstanding this problem, the question remains: if you have a set of preferences, can you find a utility function that represents them? If we restrict ourselves to monotonic preferences that have indifference curves, the answer is yes.

Creating a utility function from indifference curves and the 45^o line

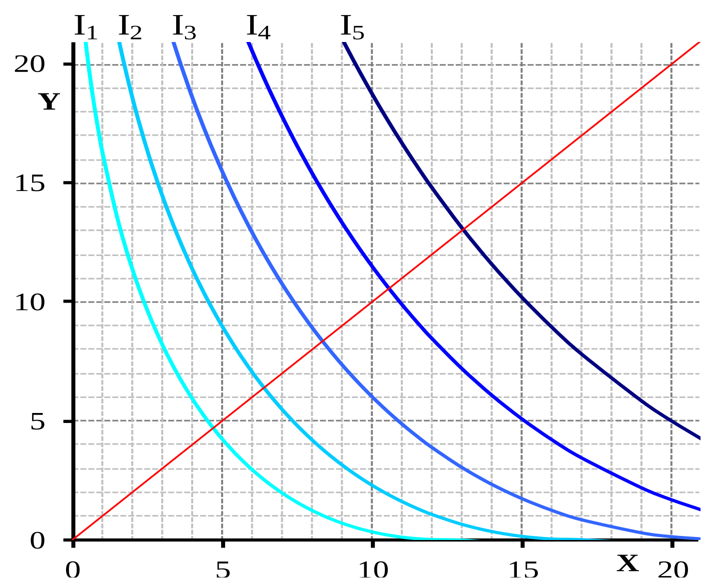
This analysis assumes the existence of nondegenerate indifference curves. The proof of this assertion is based on the observation that any point A = (x_o,y_o) has an indifference curve running through it. Monotonicity requires that the indifference curve must be in the 2nd or 4th quadrants with respect to A. If (x_o,y_o) has x_o> y_o, then the bundle is below the 45^o line (x = y) and if $x_o < y_o$, then the bundle is above the 45° line. In either event, the indifference curve containing A must eventually cross the 45° line. If we define the function as the value of x where this occurs, then we have successfully attached a numerical “score” to each bundle that satisfies all of the properties of a utility function. Recall that a utility function represents preferences if it associates smaller numbers with less preferred bundles, and larger numbers with more preferred bundles. Therefore, we have created a utility function to represent an indifference map: all that is required are indifference curves that are monotonic.

Students are sometimes reticent to call this a utility function because it is based on the geometry required by monotonicity rather than an algebraic function. A utility function need not be algebraic; it simply has to connect bundles to numbers in a way that preserves the order of preferences over bundles. The strategy proposed above does just that.

If the bundle A = (x_o,y_o) has $x_o > y_o$, then there must be a bundle in the 2nd quadrant relative to (x_o,y_o) that is indifferent to (x_o,y_o) and is on the 45° line. If $x_o < y_o$, that bundle must be in the 4th quadrant relative to (x_o,y_o). Call this bundle $(x^*,x^*) \sim (x_o,y_o)$. Define U(A) = $U(x_o,y_o) = x^*$. For example, in Figure 1 indifference curve I_5 goes through the bundle (20,5) and that indifference curve crosses the 45° line at $x^* = 13.0$ therefore U(20,5) = 13.0.

Question. What are the utility levels associated with the indifference curves $I_1 - I_4$ in Figure 1 given this utility function? (See end of article for answer.)

Figure 1.Creating a utility function from indifference curves.



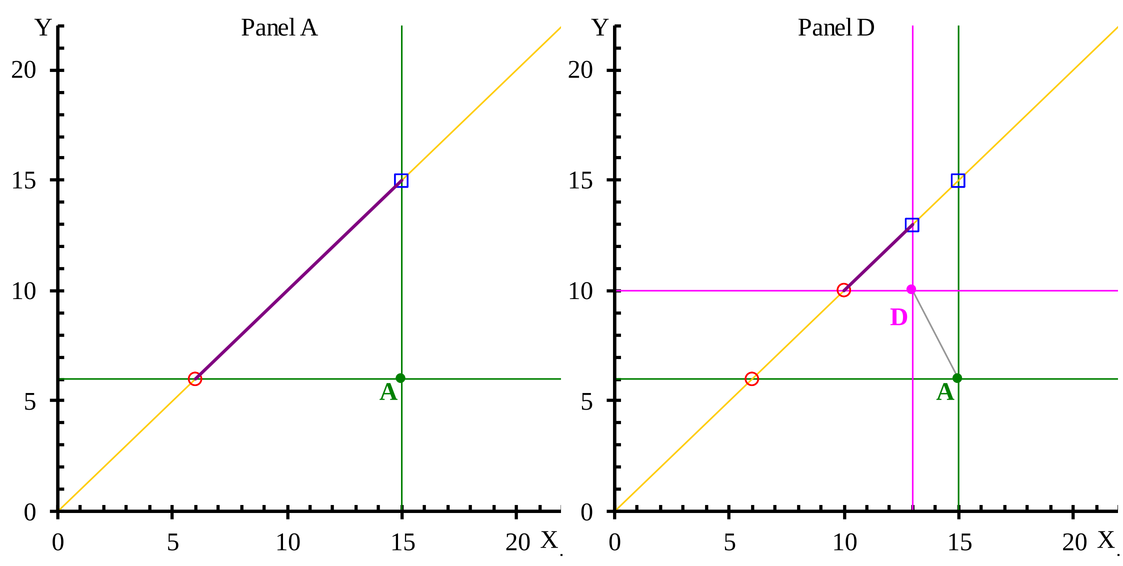
Define U (x_0,y_0) as the x value on the 45° line where $(x^*,x^*) \sim (x_o,y_o)$. For example, U(20,5)= U(16,9) = 13.0.

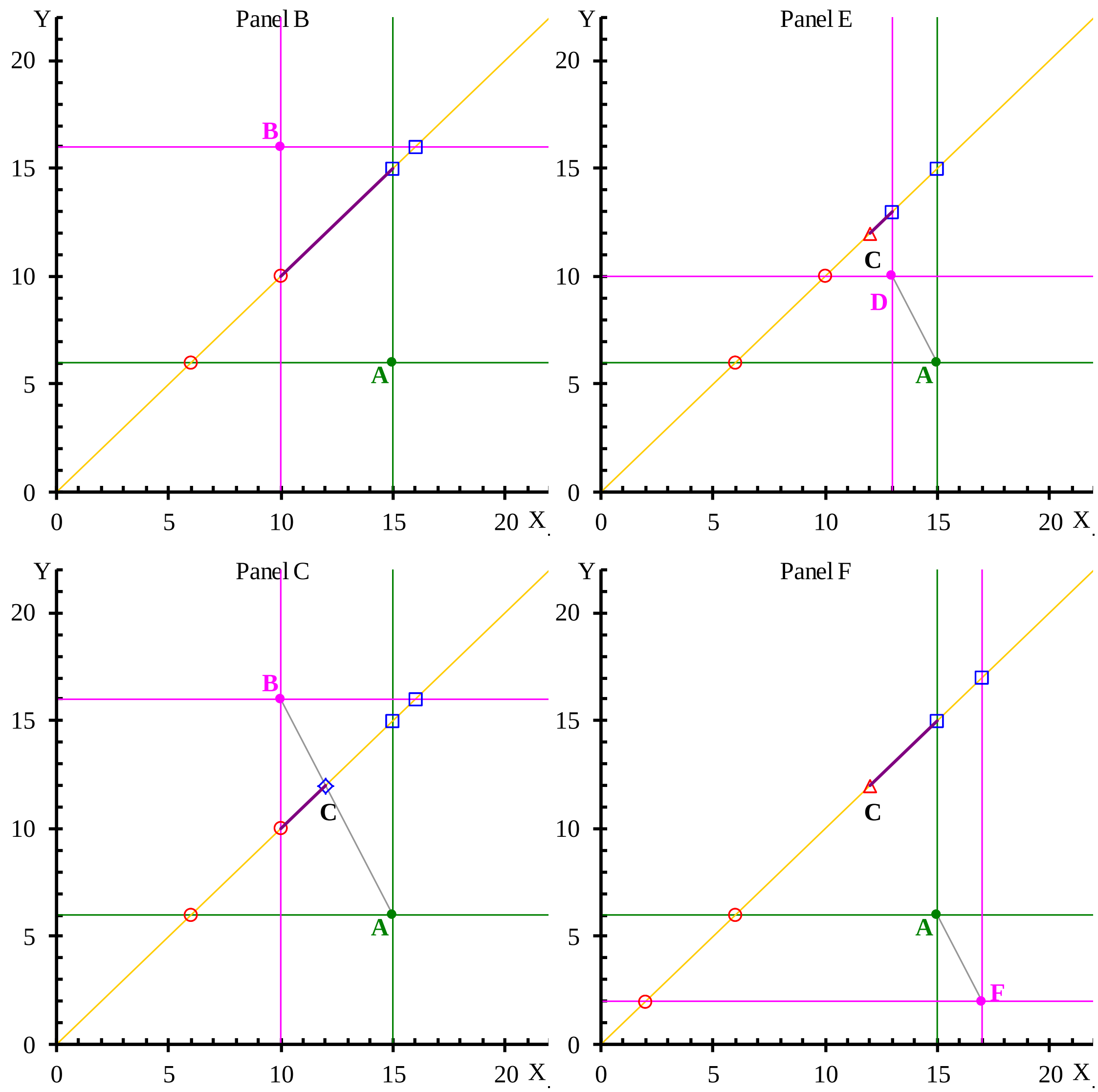
Even if we do not have sufficient information to create a utility function (for example, we do not have an indifference map of the individual's preferences), we can place bounds on a utility function using the methodology proposed for Figure 1. In the absence of indifference curves we can narrow down the range of utility level values we would attach to a given bundle by using monotonicity and convexity.

Suppose bundle **A** = (15,6). This bundle is below the 45° line in Figure 2, so there must be a point in the 2nd quadrant relative to **A** that is both on the indifference curve through **A** and on the 45° line. Put another way, all bundles in the first quadrant are better than **A** (with the possible exception of bundles on the border of this quadrant). The MAXIMUM value that the utility function can attach to bundle **A** is 15, the point on the 45° line associated with the border between the 1st and 2nd quadrants relative to **A**. The **blue square** at the bundle (15, 15) in Figure 2A denotes this maximum.

A **blue square** in Figure 2 represents the upper bound on the utility level implied by the coordinates of a given bundle. This bound is due to monotonicity. The maximum would occur if, for example, **A** was on the vertical part of a perfect complement indifference curve with vertex at the bundle (15,5). **A** dominates all bundles in the 3rd quadrant relative to **A**, therefore, the MINIMUM value that the utility function can attach to bundle **A** is 6, the point on the 45° line associated with the border between the 2nd and 3rd quadrants relative to **A**. The **red circle** at the bundle (6,6) in Figure 2A denotes this minimum. A **red circle** in Figure 2 represents the lower bound on the utility level implied by the coordinates of a given bundle. This bound is due to monotonicity.

Figure 2. Bracketing utility with monotonicity and convexity





The symbols on the Y = X line that bracket utility are based on monotonicity & convexity using following symbology: Monotonicity, □ upper bound, **o** lower bound; Convexity,◇ upper bound,∆ lower bound.

The minimum would occur if, for example, **A** was on the horizontal part of a perfect complement indifference curve with vertex at the bundle (4,6). Monotonicity has bracketed the utility level associated with bundle **A** to be somewhere between 6 and 15. This is represented on the graph by the **purple segment** on the 45^{o} line between the lower bound of 6 (the red circle) and the upper bound of 15 (the blue square) in Panel 2A.

The **purple segment** on the 45^{o} line depicts the range of possible utility levels that may be attached to a bundle. The purple segment depicts the range that the utility level has been bracketed to by the known information.

The bottom of the segment is the highest lower bound and the top of the segment is the lowest upper bound implied by the known information. (In Panel 2A, the known information is the location of a single bundle.) The other five panels examine what happens when a second indifferent bundle is discovered.

As soon as we find a second bundle, **B**, that is indifferent to **A**, **B** ~ **A**, we can further restrict the range of possible utility levels associated with bundle **A**. Consider bundle **B** = (10,16) in Panel 2B. The blue square represents the upper bound and the red circle represents the lower bound (as above with point **A**). Monotonicity reduces the range of possible utility levels with the discovery of **B** ~ **A** because the lower bound has increased to 10 from 6. The lower bound would be binding (i.e. U(**A**) = 10) if, for example, **A** and **B** were on the same perfect complement indifference curve with vertex at (10,). The upper bound remains 15 because **B**'s upper bound is higher than **A**'s (16 > 15). Monotonicity has reduced the range for the utility level associated with bundles **A** ~ **B** to the purple segment between 10 and 15 in Panel 2B.

Convexity requires that every bundle between **A** and **B** is at least as good as **A** or **B**. This further reduces the range of possible utility levels as we see in Panel 2C. The **blue diamond** at **C** = (12,12) is 60% of the way from **A** to **B** along the line segment **AB**.

The **blue diamond** in Figure 2 represents an upper bound on the utility level implied by convexity. This occurs when the two bundles are on the opposite side of the 45^o line.

Since bundle **C** is at least as good as **A** or **B**, convexity has reduced the maximum utility level U(**A**) to 12 from 15. This would be the utility level if **A** and **B** were on the same perfect substitutes indifference curve. (This indifference curve would have MRS = 2.) Convexity plus monotonicity has reduced the range for the utility level associated with bundles **A** ~ **B** to the purple segment between 10 and 12.

Convexity reduced the range of utility levels in Panel 2C because **A** and **B** were on opposite sides of the 45^o line. Interestingly, convexity will also reduce this range when the two indifferent bundles are on the same side of the 45^o line. It does so by increasing the **minimum** possible utility level (rather than decreasing the maximum possible utility level as in Panel 2C). This is the scenario explored in Panels 2D – 2F. Panels 2D and 2E examine the addition of a second bundle, **D** = (13,10) in the 2^{nd} quadrant relative to **A**. Since **D** is closer to the 45^o line, monotonicity directly reduces the range of possible utility levels associated with bundles **A** ~ **D** to the purple segment from 6 to 15 in Panel 2A to 10 to 13 in Panel 2D.

Convexity can further reduce the range of possible utility levels attached to **A** ~ **D**. In this instance, convexity increases the lower bound from 10 to 12. The convexity "implied" lower bound is represented by the **red triangle** in Panel 2E at point **C =** (12,12).

The **red triangle** in Figure 2 represents a lower bound on the utility level implied by convexity. This occurs when the two bundles are on the same side of the 45^o line.

Unlike the red circles and blue squares, there is either a red triangle or a blue diamond when there are two indifferent bundles. The determining factor in this instance is whether the two bundles are on opposing or the same side of the 45^o line. To understand why the lower bound has increased, imagine that this were not the case. For example, imagine that (11,11) ~ **A** ~ **D**. Convexity would require that bundles between (11,11) and **A** are at least as good as (11,11) or **A**. This would mean that **D** is strictly better than (11,11) or **A** because there are bundles between (11,11) and **A** that are in the 3rd quadrant relative to **D**. Further, there are bundles that are on the interior of that quadrant – these bundles include less of both goods. One such bundle is (12.6,9), 60% of the way from point **A** = (15,6) to (11,11). Monotonicity requires that **D** would strictly dominate such bundles. Transitivity requires that **D** would strictly dominate the endpoint bundles, (11,11) and **A**, as well. This contradicts the initial assumption of indifference. Therefore, convexity has increased the lower bound to point **C**. Convexity therefore further reduces the range of possible utility levels associated with bundles **A** ~ **D** to the purple segment from 10 to 13 in Panel 2D to 12 to 13 in Panel 2E.

This also can be thought of in terms of MRS between points. Once two indifferent bundles are discovered, an implicit exchange rate between those bundles is created (this is simply (-)$\Delta y/\Delta x$ between the two bundles). Given bundles **A** and **D** in Panel 2E, an MRS between **A** and **D** is set up (in this instance that MRS = 2). "Set up" does not mean that the MRS must equal 2 over the entire range from **A** to **D**. Convexity and monotonicity do imply that, given **A** ~ **D** with **D** in the 2nd quadrant relative to **A**, the MRS at **D** must be at LEAST 2 and the MRS at **A** must be at MOST 2. If this was not the case, convexity would have been violated. Therefore, the MRS to the left of **D** along the indifference curve through **A** and **D** must be at least 2 if preferences are convex. (The reverse is also true; bundles to the right of **A** on the indifference curve can be no larger than the MRS between **A** and **D** if preferences are convex and monotonic.)

Finally, Panel 2F shows that a second point reduces the range even if that point is further away from the 45^o line than the initial bundle. Suppose point **F** = (17,2) ~ **A**. The addition of **F** does not reduce the range of utility levels due to monotonicity. However, convexity does cause the range to decrease for the same reason as in Panel 2E. Convexity implies that the minimum increase from 6 to 12 because the **A** to **F** MRS = 2 implies that the MRS to the left of **A** along the indifference curve through **A** and **F** must be at least as large as 2. Therefore, convexity reduces the range of possible utility levels associated with bundles **A** ~ **F** to the purple segment from 6 to 15 in Panel 2A to 12 to 15 in Panel 2F.

The examples in Panels 2B-F all use an MRS of 2 between bundles to minimize the complexity of the discussion. The Excel file allows you to relax this assumption by varying the location of both bundles over the range (0,0) to (20,20). The only restriction in the Excel file is that the second point, the pink bundle in the diagram, must be in the 2nd or 4th quadrant relative to the first point, the green bundle in the diagram. If this were not the case, monotonicity would be violated. The file also allows you to click on and off the various lower and upper bounds associated with monotonicity and convexity, line segments and shifted axes at each bundle. Together, the controls allow you to explore the bracketing phenomenon discussed above in greater detail.

The relation between bracketing utility and revealed preference

Revealed preference theory is based on the idea that as an individual chooses bundles subject to a budget constraint they reveal information about the individual's underlying preferences. As the individual's budget constraint changes, new information is revealed about underlying preferences. In this way, we are able to recover or trap an individual's preference map (indifference curve). This recovery process does not lead to a unique solution, but it provides bounds on that solution. The conceptual experiment presented in the bracketing utility discussion provides a similar analysis on the utility side. As we discover a second bundle that is indifferent to an initial bundle, we are able to place bounds on the level of utility achieved by those individual bundles. The process can be refined to adding a third indifferent bundle. The questions below provide guidance on this refinement process. Answers are provided here, the Excel file includes the questions but not the answers.

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Suggested questions

1. Suppose you are indifferent between 3 bundles, **A** = (4,10), **B** = (7,4) and **C** = (13,1). You have monotonic and convex preferences and would like to bracket the utility level that these bundles would achieve if you used the utility function described above (the utility attained by a bundle is equal to x* in the bundle (x*,x*) which has the property: (x*,x*) ~ **A** ~ **B** ~ **C**). In this instance, does the addition of a third indifferent bundle make the range of possible values decrease? Explain. (You can do this by careful use of the Excel file but it is perhaps easier if you just use graph paper and follow the strategy put forward in the discussion of Figure 2.)

Answer 1: *The combined range is 5-6, but the range with any two is larger:*

AB *is 4-6,* ***AC*** *is 4-7 and* ***BC*** *is 5-7.*

The addition of a third indifferent bundle does help reduce the range of possible utility values.

2. Suppose you are indifferent between 3 bundles, **A** = (4,10), **B** = (8,2) and **C** = (13,1). You have monotonic and convex preferences and would like to bracket the utility level that these bundles would achieve if you used the utility function described above (the utility attained by a

bundle is equal to x* in the bundle (x*,x*) which has the property: (x*,x*) ~ **A** ~ **B** ~ **C**). In this instance, does the addition of a third indifferent bundle make the range of possible values decrease? Explain. (You can do this by careful use of the Excel file but it is perhaps easier if you just use graph paper and follow the strategy put forward in the discussion of Figure 2.)

Answer 2: *The combined range is 4-6, the same range as* ***AB*** *is 4-6. AC is 4-7 and* ***BC*** *is 3-8.*

The addition of a third indifferent bundle does NOT help reduce the range of possible utility values when that bundle is bundle ***C****.*

3. Explain why the answers to Questions 1 and 2 are different.

Hint: Consider how ***A****,* ***B****, and* ***C*** *differ in the two questions.*

Answer 3: *Notice that* ***A*** *and* ***C*** *are the same for both problems. The sole difference is in the location of* ***B****.*

The implied ***BC*** *min due to convexity (5) is larger than* ***A****'s x value (of 4) in the first instance (5 > 4) so the range is restricted. In the second instance, the implied* ***BC*** *min due to convexity is less than* ***A****'s x value (3 < 4).*

Answer to the question for Figure 1: *The utility function U(x,y) associated with indifference curves* I_1 – I_4 *passing through the points (3,8), (6,7), (10,6) and (15,5) in Figure 1 has values U(3,8) = 4.7, U(6,7) = 6.4, U(10,6) = 8.4, and U(15,5) = 10.6.*

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