5-19-2013

Classification of Symbolic Dynamics for One-Dimensional Dynamical Systems With Overlapping Regions

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CLASSIFICATION OF SYMBOLIC DYNAMICS FOR ONE-DIMENSIONAL DYNAMICAL SYSTEMS WITH OVERLAPPING REGIONS

By

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Submitted in partial fulfillment of the requirements for departmental honors in Mathematics
Dickinson College, 2012-2013

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May 15, 2013
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When studying a dynamical system, it is common to partition the space (or a subset of the space) into a finite number of disjoint regions. Associated to each orbit is its itinerary, the sequence of regions it passes through. If the regions in the space overlap, a single orbit can have multiple itineraries. Hence, the itineraries are ambiguous. In order to study such systems, we need a bank of examples. We can represent the example via a directed graph (the transition graph from the dynamical system) and an undirected graph (the intersection graph from the intervals). We will discuss which pairs of transition and intersection graphs can be realized by continuous one-dimensional dynamical systems (on $\mathbb{R}$ and on $S^1$). We also count the numbers of possible transition graphs in the case of disjoint intervals. Moreover, we can generate a realization in the form of a piecewise linear function for every such pair of intersection graph and transition graph. We will use techniques from graph coloring, combinatorics, algorithms, and dynamical systems theory.
ACKNOWLEDGMENTS

I would like to express my great appreciation to my thesis advisor Professor David Richeson for his patient guidance and valuable assistance in this paper. I would also like to thank Professor Lorelei Koss and Professor Barry Tesman for constructive suggestions throughout this research work.
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1. INTRODUCTION

A dynamical system is used to describe how a state evolves into the next over time. For any function \( f : X \to X \), the orbit of \( x_0 \in X \) is the sequence \( (x_k) \) where \( x_{k+1} = f(x_k) \). For convenience, we define \( f^2(x) = f(f(x)) \) and \( f^n(x) = f(f^{n-1}(x)) \) for \( n \geq 3 \).

One approach to studying a dynamical system is through symbolic dynamics. We divide the space (or a subset of the space) into a finite number of regions \( N_1, \ldots, N_n \). For example, we may create a finite number of intervals in a one-dimensional dynamical system or a finite number of rectangles in a two-dimensional dynamical system. Instead of studying the exact values of the sequence of \( f^n(x) \), we look at the sequence of regions that contain the points \( f^n(x) \). The sequence \( (a_0, a_1, \cdots) \) is an itinerary for \( x \) provided that \( f^k(x) \in N_{a_k} \) for \( k = 0, 1, \ldots \). Properties of the itinerary of the orbit can give us information about the underlying dynamical system. Traditionally, the regions are non-overlapping or only intersect along the boundaries. We are looking at possible nontrivial intersections and trivial ones in which we do not know the behavior.

Suppose we have a one-dimensional dynamical system: a continuous function \( f : \mathbb{R} \to \mathbb{R} \) and a finite number of closed intervals \( I_1, I_2, \ldots, I_n \subset \mathbb{R} \). Construct a directed graph as follows. There are \( n \) vertices denoted \( 1, 2, \ldots, n \) and an edge from vertex \( i \) to vertex \( j \) if \( f(I_i) \supseteq I_j \); in this case we say \( I_i \) \( f \)-covers \( I_j \), and we write \( I_i \to I_j \). For simplicity, sometimes we say “covers” instead of “\( f \)-covers”. We will make the standing assumption that every interval \( f \)-covers at least one interval. Hence, in the digraph, every vertex has at least one outgoing edge. We call this a transition graph.

**Example 1.1.** Figure 1.1 shows a one-dimensional dynamical system with 3 disjoint intervals. We see that \( f(I_1) \supseteq I_1 \), \( f(I_2) \supseteq I_1 \cup I_2 \cup I_3 \), and \( f(I_3) \supseteq I_2 \). So, \( I_1 \) covers itself,
FIGURE 1.1. An example with 3 intervals and its related graph

$I_2$ covers all three intervals, and $I_3$ covers $I_2$. The associated transition graph is shown on the right.

The power of symbolic dynamics can be illustrated by the following famous theorems: the Li-Yorke theorem and its generalization, Sharkovskii's theorem. Li and Yorke's 1975 paper "Period three implies chaos" drew a lot of attention. However, it turned out that O. M. Sharkovskii had proved a stronger statement in 1964. First, we need some definitions.

**Definition 1.2.** The point $x$ is a **periodic point of period** $n$ if and only if $x = f^n(x)$. If $n$ is the smallest such integer, it is called **least period**.

**Definition 1.3.** The point $x$ is **eventually periodic** with period $n$ if and only if $f^p(x) = f^{p+n}(x)$ for any $p$ greater than some $N$.

The Li-Yorke theorem states:

**Theorem 1.4 ([LY75]).** Let $J$ be an interval and let $F: J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy
\[ a < b < c \text{ (or } d \geq a > b > c) \]. For every \( k = 1, 2, \cdots \), there is a periodic point in \( J \) having period \( k \).

**Corollary 1.5.** Let \( J \) be an interval and let \( F : J \rightarrow J \) be continuous. Assume there is a periodic point \( a \in J \) with period 3. There exist periodic points in \( J \) with period \( k \) for every \( k \).

**Proof.** According to Theorem 1.4, if \( d = a \), then \( a, b, c \) form a period three orbit. Then there exist periodic points in \( J \) with period \( k \) for every \( k \). \( \square \)

Hence, period three implies chaos.

In order to state Sharkovskii’s theorem, we must introduce a new ordering on the positive integers using the symbol \( \gg \) called the Sharkovskii ordering.

**Definition 1.6.** Sharkovskii ordering:

\[
3 \gg 5 \gg 7 \gg \cdots \gg 2 \cdot 3 \gg 5 \gg 2 \cdot 7 \gg \cdots \gg 2 \cdot 2 \cdot 3 \gg 2 \cdot 5 \gg 2 \cdot 2 \cdot 7 \gg \cdots \gg 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \gg 2 \cdot 2 \cdot 2 \cdot 5 \gg 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 \gg \cdots \gg 2^k \cdot 3 \gg 2^k \cdot 5 \gg 2^k \cdot 2^2 \cdot 7 \gg \cdots \gg 2^n \cdot 3 \gg 2^n \cdot 5 \gg 2^n \cdot 2^2 \cdot 7 \gg \cdots \gg 2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \gg 1.
\]

**Theorem 1.7** ([Šar64]). Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function from an interval \( I \) into the real line. Assume \( f \) has a point of least period \( n \) and \( n \gg k \). Then \( f \) has a point of least period \( k \).

While the Li-Yorke theorem says that if there is a point of period 3 then there exist points of every period, Sharkovskii’s theorem gives a stronger conclusion: given a point of any period, the Sharkovskii ordering tells us that there exist points with some other periods.
In the proofs of both theorems the domain is divided into intervals and the corresponding transition graph is drawn. Then they prove that given any path through the graph, there is a point whose itinerary falls into these intervals in the exactly same order. They do a little more work to show that periodic paths correspond to periodic orbits.

Next, we use undirected graphs to describe how intervals overlap.

Suppose there are \( n \) closed intervals \( I_1, I_2, \ldots, I_n \) in \( \mathbb{R} \). Construct a graph as follows. There are vertices 1, 2, \ldots, \( n \) representing \( n \) intervals. There is an edge (which we represent as a dashed line) between vertices \( i \) and \( j \) if and only if \( I_i \cap I_j \neq \emptyset \). More generally, an intersection graph, or an interval graph, is any graph that can be obtained in this way.

![Figure 1.2. An example of overlapping intervals](image)

**Example 1.8.** Figure 1.2 shows an example of four intervals and the associated interval graph. We note that there are two types of intersection. One is a partial overlap such as \( I_1 \) and \( I_2 \), for which \( I_1 \cap I_2 \neq \emptyset \), \( I_1 \cap I_2 \neq I_1 \) and \( I_1 \cap I_2 \neq I_2 \). The other is complete containment, such as \( I_2 \) and \( I_3 \), for which \( I_3 \subset I_2 \).

In this paper we also need to consider the structure of the intervals and the dynamical system at the same time. Given a collection of intervals \( \{I_i\} \) and a dynamical system...
Figure 1.3. An example of a one-dimensional dynamical system with overlapping intervals $f : \cup I_i \to \mathbb{R}$, it is useful to superimpose the intersection graph and the transition graph discussed above.

**Example 1.9.** Figure 1.3 shows a one-dimensional dynamical system with overlapping intervals and the associated graphs.

In rest of the paper, we study what information we can obtain from the transition graphs if intervals overlap. Also, we give a partial answer to the question: given such a pair of superimposed graphs, does there exist a set of intervals $I_1, I_2, \ldots, I_n$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ that realizes them? If so, we show how to construct them. This investigation is interesting because it is inspired by the work of Fabio Drucker, David Richeson, and Jim Wiseman (see [DRW]). They studied the entropy of symbolic dynamical systems with overlapping regions. But they found that it was sometimes hard to come up with good examples. This thesis provides a way to identify when we can realize a given pair of transition graph and intersection graph, and construct realizations of intervals and functions.
In Section 2, we explore some nice properties of interval graphs. In Section 3, we prove that paths through the transition graph imply actually orbits with those itineraries. It is a general conclusion and has no requirement on the structure of intervals. In Section 4, we classify possible and impossible transition graphs for different structures of intervals on the real number line. In Section 5, we investigate when we can find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and intervals $I_1, I_2, \ldots, I_n$ that realize a given pair of transition graph and intersection graph. First, we look for a set of intervals that has the given intersection graph. Next, we try to determine if the transition graph can be realized by a continuous function. In Section 6, we present techniques that we can use to construct function realizations as piecewise linear functions. In the last section, we consider the other case of one-dimensional space: a circle. We apply similar analysis to classify transition graphs and discuss the algorithm that generates circular-arc graphs.
In the introduction section, we discussed how to construct an interval graph from a set of intervals. We further investigate the properties of intervals graphs in this section. Since there are different structures of overlapping, first we restrict our attention to partial overlapping.

**Definition 2.1.** A *proper interval graph* is an interval graph that has an interval representation in which no interval is completely contained in any other intervals.

![Figure 2.4. Four partially overlapping intervals](image)

We will look at special cases of proper interval graphs. Let each interval $I_k$ only overlap with its neighbors. An example is shown in Figure 2.4. Notice that in this restricted partial overlapping interval case, dashed lines only exist between two consecutive intervals.

There is another type of partial overlap, as shown in Figure 2.5, where $I_3$ partially overlaps with not only its neighbor $I_2$ but also $I_1$. But no interval is contained in another.

**Definition 2.2.** An *n-staircase* is a collection of intervals $I_1, I_2, \ldots, I_n$ where $I_i \cap I_j \neq 0$ and $I_i \not\subset I_j$ for all $i \neq j$. 

7
**Definition 2.3.** A *clique* is a collection of vertices in a graph, in which every pair of vertices in the collection is joined by an edge.

Clearly, if there is an $n$-staircase structure, there will be an $n$-clique in the overlapping graph. However, the converse of the statement is not true: a 3-clique in the overlapping graph can also be a result of $I_3 \subset I_2 \subset I_1$. But in any proper interval graph, there is a $n$-clique if and only if there is an $n$-staircase structure.

![Figure 2.5. Three-staircase structure intervals](image)

**Definition 2.4.** Let $G = (V, E)$ be a graph. A *path* in $G$ is a sequence $v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_{t+1}$ where each $v_i \in V$, each $e_i \in E$, where $e_i = \{v_i, v_{i+1}\}$ if $G$ is undirected and $e_i = (v_i, v_{i+1})$ if $G$ is directed. A path is a *cycle* if $v_{t+1} = v_1$ and the vertices $v_1, v_2, \ldots, v_t$ are distinct.

**Definition 2.5.** A *chord of a cycle* is an edge between nonconsecutive vertices of the cycle. A graph is *chordal* if and only if every cycle large enough to have a chord has a chord.

**Definition 2.6.** Three distinct vertices $x, y$ and $z$ form an *asteroidal triple* when for each pair of vertices in $\{x, y, z\}$, there is a path joining them, with no vertex on the path adjacent to the third.

**Example 2.7.** The vertices $a, b,$ and $c$ in Figure 2.6 form an asteroidal triple.
The following theorem gives us a nice way to check whether a graph is an interval graph.

**Theorem 2.8 ([LB63]).** A graph is an interval graph if and only if it is chordal and has no asteroidal triple.

Recall that any graph isomorphic to the graph in Figure 2.7 is $K_{1,3}$.

![Figure 2.6. Asteroidal Triple](image)

**Figure 2.6. Asteroidal Triple**

The following theorem provides a way to identify whether an interval graph is proper.

**Theorem 2.10 ([Rob69]).** A graph is a proper interval graph if and only if it is an interval graph that does not contain an induced subgraph isomorphic to $K_{1,3}$.

![Figure 2.7. $K_{1,3}$](image)

**Figure 2.7. $K_{1,3}$**

**Definition 2.9.** $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$, and for any $a, b \in V'$, \{a, b\} $\in E'$ if and only if \{a, b\} $\in E$. 

The following theorem provides a way to identify whether an interval graph is proper.
Example 2.11. The graph in Figure 2.8 is chordal and does not have asteroidal triple. So it is an interval graph by Theorem 2.8. Moreover, according to Theorem 2.10, the graph is not a proper interval graph because the subgraph induced by \( \{v_2, v_3, v_5, v_6\} \) is \( K_{1,3} \).

![Figure 2.8. A nonproper interval graph](image)

We will use the idea of graph coloring to study possible and impossible transition graphs. It is often useful to study the colorings of a graph, i.e. we color the vertices different colors according to given rules. Graphs that satisfy the coloring rules are possible graphs and otherwise, impossible graphs. In our case, we define the color of a vertex to be the outset of the vertex.

Example 2.12. Figure 2.9 shows an example of a one-dimensional dynamical system with four closed intervals that share endpoints. Moreover, \( I_1 \) covers itself, \( I_2 \) covers \( I_3 \) and \( I_4 \), \( I_3 \) covers \( I_4 \), and \( I_4 \) covers \( I_2 \). Notice that since \( I_1 \) covers \( I_1 \) and \( I_2 \) covers \( I_4 \), \( I_2 \) must also cover \( I_3 \). Using the coloring idea, we say that vertex 1 has color \{1\}, vertex 2 has \{3,4\}, vertex 3 has \{4\}, and vertex 4 has \{2\}. 
3. INTERVAL ITINERARIES IMPLY ACTUAL ORBITS OF POINTS

In symbolic dynamics, we can easily obtain an itinerary from an orbit \((f^n(x))\). Is the converse true? The following theorem states conditions under which an interval itinerary corresponds to an actual orbit \((f^n(x))\). Recall that the arrow symbol \((\rightarrow)\) means "\(f\)-covers."

**Theorem 3.1.** Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function and \(I_1, I_2, \ldots, I_n \subset \mathbb{R}\) be a finite number of closed intervals. If there exists a sequence \(J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow \cdots\) where each \(J_i = I_k\) for some \(k\), then there exists a point \(c_0 \in J_0\) such that \(f^i(c_0) \in J_i\) for every \(i\).

Notice that in the hypothesis of the theorem, we do not require the closed intervals to be disjoint. To prove this theorem, we need the following lemma that is proved in [Rob99, p.64].

**Lemma 3.2.** If \(I\) and \(J\) are closed intervals and \(f(I) \supset J\) then there exists a closed subinterval \(K \subset I\) such that \(f(K) = J\).

By Lemma 3.2, we can prove the following statement.
Lemma 3.3. Given the hypotheses of Theorem 3.1, there exists a closed subinterval $K_t \subset J_0$ such that for $i = 1, 2, \ldots, t$, $f^i(K_t) \subset J_i$ and $f^t(K_t) = J_t$.

Proof. This is a proof by induction. The induction statement $(S_t)$ is as follows: there exists a closed subinterval $K_t \subset J_0$ such that for $i = 1, 2, \ldots, t$, $f^i(K_t) \subset J_i$ and $f^t(K_t) = J_t$.

By Lemma 3.2, $(S_1)$ is true. Now suppose $(S_t)$ is true for some $t \geq 1$. That is, there exists a closed subinterval $K_t \subset J_0$ such that for $i = 1, 2, \ldots, t$, $f^i(K_t) \subset J_i$ and $f^t(K_t) = J_t$. Therefore, $f^{t+1}(K_t) = f(f^t(K_t)) = f(J_t) \supset J_{t+1}$. By Lemma 3.2 again, there exists a closed subinterval $K_{t+1} \subset K_t$ such that $f^{t+1}(K_{t+1}) = J_{t+1}$. Hence, $(S_t)$ is true for all $t \geq 1$.

Proof of Theorem 3.1. Let $K_t$ be sets defined in Lemma 3.3. Notice that $K_1 \supset K_2 \supset K_3 \supset \cdots$ and $K_n$ is a closed interval for all $n \in \mathbb{N}$. By the nested interval property, there exists $c_0 \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. By Lemma 3.3, since $c_0 \in K_0$ and $f^t(K_0) \subset J_t$ for all $t$, $f^t(c_0) \in J_t$ for all $t$.

The following corollary follows immediately from Theorem 3.1.

Corollary 3.4. Suppose $I_1, I_2, \ldots, I_n \subset \mathbb{R}$ are closed intervals, $f : \bigcup I_i \to \mathbb{R}$ is continuous, and $G$ is the associated transition graph. Given any path in $G$, there exists a point $c$ with that itinerary.

Example 3.5. Let $I_1 = [0, 3]$, $I_2 = [2, 4]$, and $I_3 = [5, 6]$. Figure 3.10 shows a continuous piecewise function and the associated graph. Notice that $(1, 2, 3, 1, 1, 1, \ldots)$ is a path through the graph. By Theorem 3.1, there exists a point with the itinerary $I_1 \to I_2 \to$
In particular, \( x = \frac{2}{3} \) has the orbit \((\frac{2}{3}, 2, 6, 0, 0, 0, \ldots)\) and hence the itinerary \((1, 2, 3, 1, 1, 1, \ldots)\).

![Graph]

**Figure 3.10.** A piecewise function

**Example 3.6.** In Figure 1.3, \( I_1 \rightarrow I_1 \rightarrow I_3 \rightarrow I_4 \rightarrow I_2 \rightarrow I_4 \rightarrow I_4 \rightarrow \cdots \) is a path in the transition graph. Thus, there must exist a point with the itinerary \((1, 1, 3, 4, 2, 4, 4, 4, \ldots)\).

Note that there might be two or more points that have the same itinerary. Hence, we cannot simply assume that periodic itineraries correspond to periodic orbits. However, under certain conditions (which we will not discuss here), every itinerary corresponds to one and only one point’s orbit. In this case, periodic itineraries correspond to periodic points.

**Theorem 3.7.** Let \( f : \bigcup I_i \rightarrow \mathbb{R} \) be a continuous function and suppose every itinerary corresponds to a unique orbit, then every periodic itinerary \((1, 2, \ldots, n, 1, 2, \ldots, n, \ldots)\) corresponds to a point of period \( n \). If the \( I_i \)'s are mutually disjoint then the point has least period \( n \).
Proof. Let \( a \in J_1 \) have the itinerary \((1, 2, \ldots, n, 1, 2, \ldots, n, \ldots)\), then \( f^n(a) \) also has the same itinerary. Because every itinerary corresponds to a unique point, it must be true that \( a = f^n(a) \). Hence, \( a \) is a point of period \( n \). \( \square \)
In this section, we focus on classifying and counting possible and impossible associated transition graphs. We define "possible" and "impossible" in terms of whether we can find a set of intervals and a continuous dynamical system \( f : \mathbb{R} \to \mathbb{R} \) that realizes the intersection graph and the transition graph. Recall our assumption of the transition graph: every vertex has at least one outgoing edge. We will consider the cases in which these intervals are disjoint, in which they only intersect with the neighboring intervals, in which their endpoints map to other endpoints, and in which there are staircase structures and containments.

4.1. Case I: (Disjoint intervals). First, we consider the case that the intervals \( I_1, \ldots, I_n \) are disjoint. Without loss of generality, say \( I_k = [a_k, b_k] \) for \( 1 \leq k \leq n \) where \( a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \). In this case, the corresponding interval graph consists of \( n \) vertices and no edges as shown in Figure 4.11.

![Interval graph of case I](image)

**Figure 4.11.** Interval graph of case I

Notice that since there are gaps between any two consecutive intervals \( I_k \) and \( I_{k+1} \), the intervals that \( I_{k+1} \) covers are independent of those that \( I_k \) covers. There is only one restriction: because \( f \) is continuous, \( I_k \) must cover a sequence of consecutive intervals. For example, if \( I_k \) covers \( I_i \) and \( I_j \) with \( i < j \), \( I_k \) covers all the intervals between \( I_i \) and \( I_j \). In other words, it is impossible for \( I_1 \) to cover \( I_1 \) and \( I_3 \), but not \( I_2 \).
It is also easy to count the number of graphs that can be obtained under our hypotheses.

From now on, we will call them *possible graphs*. Each interval must cover at least one interval, but may cover a sequence of consecutive intervals. If an interval covers exactly one interval, the corresponding edge can point to any one of the *n* nodes. So, there are \( \binom{n}{1} = n \) possible arrangements for the nodes' outgoing edge. Similarly, suppose an interval covers a sequence of consecutive intervals. The number of ways this can happen is equal to the number of different ways of choosing a starting interval and an ending interval; this number is \( \binom{n}{2} \). Therefore, there are \( \binom{n}{1} + \binom{n}{2} \) possible edge arrangements for each vertex. Thus, for *n* intervals, there are \( \left( \binom{n}{1} + \binom{n}{2} \right)^n \) different possible graphs.

**Theorem 4.1.** Let \( I_1, I_2, \ldots, I_n \subset \mathbb{R} \) be disjoint closed intervals. There are \( \left[ \frac{n(n+1)}{2} \right]^n \) possible transition graphs that can be realized by a continuous function \( f : \mathbb{R} \to \mathbb{R} \), assuming each interval \( f \)-covers at least one interval.

It is clear that there are \( 2^n \) different directed graphs with *n* vertices: each of the *n* nodes can have at most *n* edges coming out of it and each edge can either exist or not exist.

There are two ways to count impossible graphs. One is to simply subtract the number of possible ones from the total number of directed graphs, which is \( 2^n - \left[ \frac{n(n+1)}{2} \right]^n \). The other approach is more complicated, and yields a different expression for this same value. As we know from counting possible graphs, there are \( \frac{n(n+1)}{2} \) possible edge arrangements for each vertex. Thus, the number of impossible arrangements for each vertex is \( 2^n - \frac{n(n+1)}{2} \). In addition, this vertex could be any of the *n* vertices, so we
multiply by \(\binom{n}{1}\). We do not care if the outgoing edges from the other \(n - 1\) vertices are possible or impossible. So we multiply by \((2^n)^{n-1}\). By fixing one vertex with impossible outgoing edges, the number of impossible graphs is \(\left[2^n - \frac{n(n+1)}{2}\right]^1(2^n)^{n-1}\binom{n}{1}\). But apparently we are over-counting the possible graphs since we did not consider the edge arrangements of the other \(n - 1\) nodes. Now we need to subtract the number of impossible graphs by fixing two vertices with impossible outgoing edges. Using a similar argument, we get \(\left[2^n - \frac{n(n+1)}{2}\right]^2(2^n)^{n-2}\binom{n}{2}\). However, we are subtracting too much. Then we need to add back the number of impossible graphs by fixing three vertices with impossible outgoing edges, which equals \(\left[2^n - \frac{n(n+1)}{2}\right]^3(2^n)^{n-3}\binom{n}{3}\). We continue this pattern according to the inclusion-exclusion principle in combinatorics. Hence, we end up with the alternating sum \(\sum_{i=1}^{n}(-1)^{i+1}\left[2^n - \frac{n(n+1)}{2}\right]^i(2^n)^{n-i}\binom{n}{i}\). From the result, it is clear that this sum equals \(2^n - \left[\frac{n(n+1)}{2}\right]^n\).

**Theorem 4.2.** Let \(I_1, I_2, \ldots, I_n \subset \mathbb{R}\) be disjoint closed intervals. There are \(2^n - \left[\frac{n(n+1)}{2}\right]^n\), or equivalently \(\sum_{i=1}^{n}(-1)^{i+1}\left[2^n - \frac{n(n+1)}{2}\right]^i(2^n)^{n-i}\binom{n}{i}\), directed graphs that cannot be realized by a continuous function \(f : \mathbb{R} \to \mathbb{R}\), assuming each interval \(f\)-covers at least one interval.

### 4.2. Case II: (Intervals overlapping only with their neighbors)

Suppose there are \(n\) closed intervals \(I_1, I_2, \ldots, I_n\) in \(\mathbb{R}\) where \(I_i \cap I_j \neq I_i\) and \(I_i \cap I_j \neq I_j\) for any \(i \neq j\), and \(I_i \cap I_j = \emptyset\) if \(|j - i| \neq 1\). That is, if any interval overlaps with another, it can only overlap with its neighbor intervals. The intersection may be a closed interval or a single point but no interval is a subset of another interval. Without loss of generality, we have
this ordering $I_k = [a_k, b_k]$ for $1 \leq k \leq n$ with $a_k < a_{k+1}$, $b_k < b_{k+1}$, and $b_k < a_{k+2}$ for any $k$. Figure 4.12 is an example of such an interval graph.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

**Figure 4.12.** An interval graph of case II

For any continuous function $f : \mathbb{R} \to \mathbb{R}$, the associated directed graph has one more restriction than it had in case I. If $I_k \cap I_{k+1} \neq \emptyset$, $f(I_k) \supset I_i \cup I_{i+1} \cup \ldots \cup I_j$, and $f(I_{k+1}) \supset I_s \cup I_{s+1} \cup \ldots \cup I_t$, then either $j + 2 \geq s$ or $i + 2 \geq t$.

**Example 4.3.** Suppose there are four closed intervals $I_1$ through $I_4$ where the interval graph is 1–2–3–4. The possible combinations of intervals each interval can cover (independent of what the previous interval covers) are \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}. Notice that $I_k \cap I_{k+1} \neq \emptyset$. Recall that we color a vertex by its outset. If $I_k$ has the color \{1\}, then $I_{k+1}$ can have any color except \{4\}. Similarly, if $I_k$ has the color \{4\}, then $I_{k+1}$ can have any color except \{1\}. We will write \{1\} $\not\leftrightarrow$ \{4\} to indicate that two neighboring vertices in the interval graph cannot have colors \{1\} and \{4\}. This is the only coloring rule in the case of 4 intervals; all other color combinations can be neighbors.

**Example 4.4.** Suppose there are five closed intervals $I_1, \ldots, I_5$ where the interval graph is 1–2–3–4–5. The coloring rules are: \{1\} $\not\leftrightarrow$ \{4\}, \{1\} $\not\leftrightarrow$ \{5\}, \{1\} $\not\leftrightarrow$ \{4, 5\}, \{2\} $\not\leftrightarrow$ \{5\}, and \{1, 2\} $\not\leftrightarrow$ \{5\}. 

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In general, we can summarize the coloring rules in this case as follows. All colors have the form \( \{I_i, I_{i+1}, \ldots, I_j\} \) for some \( i \leq j \). Then the coloring rules state that for vertices \( a \) and \( b \) with \( I_a \cap I_b \neq \emptyset \), \( \{I_i, I_{i+1}, \ldots, I_j\} \nleftrightarrow \{I_s, I_{s+1}, \ldots, I_t\} \) if and only if \( |j - s| \geq 3 \).

If there are three intervals, the possible graphs that can be obtained under our hypotheses in case II is the same as that in case I. When the numbers of intervals increases, it is difficult to obtain a general formula to compute the number of possible graphs, because what one interval can cover depends on what the previous interval covers.

4.3. **Case III: (Intervals sharing endpoints that map to other endpoints).** This is a special case of II. Suppose there is a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and there are \( n \) closed intervals \( I_1, I_2, \ldots, I_n \) in \( \mathbb{R} \) where \( I_i \cap I_j = \emptyset \) or \( I_i \cap I_j = \{p\} \) for some \( p \in \mathbb{R} \). Also assume that \( f(a_k) = a_j \) for some \( j \). That is, all endpoints are eventually periodic. Moreover, assume \( I_k = [a_{k-1}, a_k] \) for \( 1 \leq k \leq n \) and \( a_0 < a_1 < \ldots < a_n \). For any interval \( I_k = [a_{k-1}, a_k] \) with \( f(a_{k-1}) = a_i \) and \( f(a_k) = a_j \), the sequence of intervals that \( I_k \) covers must contain all intervals from \( I_{i+1} \) to \( I_j \) (including \( I_{i+1} \) and \( I_j \)) if \( j > i \), or all intervals from \( I_{j+1} \) to \( I_i \) (including \( I_{j+1} \) and \( I_i \)) if \( j < i \).

**Example 4.5.** Figure 4.13 shows an example of a continuous function \( f \) with 4 intervals for which \( a_0 \rightarrow a_4 \rightarrow a_1 \rightarrow a_3 \rightarrow a_2 \rightarrow a_0 \) forms a periodic orbit of a period of 5. In addition, \( f(I_1) \supset I_4 \), \( f(I_2) \supset I_1 \cup I_2 \cup I_3 \), \( f(I_3) \supset I_1 \cup I_2 \), and \( f(I_4) \supset I_2 \).

**Example 4.6.** Given the behavior of the endpoints we can count the number of possible graphs. Using the same 5-cycle as in Example 4.5, \( a_0 \rightarrow a_4 \rightarrow a_1 \rightarrow a_3 \rightarrow a_2 \rightarrow a_0 \), we know that \( I_1 \) must cover \( I_4 \) since \( f(a_0) = a_4 \), \( f(a_1) = a_3 \), and \( [a_3, a_4] = I_4 \). Therefore, it is possible for \( I_1 \) to cover any sequence of intervals containing \( I_4 \). The possible sequences...
Figure 4.13. An example with 4 intervals sharing endpoints that are periodic points

are \{I_1\}, \{I_3, I_4\}, \{I_2, I_3, I_4\}, and \{I_1, I_2, I_3, I_4\}. Now let’s take a look at the possible sequences of intervals that \(I_2\) can cover. Notice that \(I_2 = [a_1, a_2]\) and \(f(a_1) = a_3, f(a_2) = a_0\). Moreover, \([a_0, a_3] = I_1 \cup I_2 \cup I_3\). Hence, \(I_2\) could cover either \(\{I_1, I_2, I_3\}\) or \(\{I_1, I_2, I_3, I_4\}\). With similar arguments we can show that \(I_3\) can cover 3 possible sequences of intervals and \(I_4\) can cover 6 possible sequences of intervals. Thus, with the particular 5-cycle \(a_0 \rightarrow a_4 \rightarrow a_1 \rightarrow a_3 \rightarrow a_2 \rightarrow a_0\), there are \(4 \times 2 \times 3 \times 6 = 144\) possible associated graphs.

Additionally, if \(f(I_k) \supset I_t \cup I_{t+1} \cup \ldots \cup I_j\) and \(f(I_{k+1}) \supset I_s \cup I_{s+1} \cup \ldots \cup I_t\), it must be true that either \(j \geq s\) or \(t \geq i\). We can use the coloring idea again, but we have different rules than in case II. The rule for three intervals is \(\{1\} \not\rightarrow \{3\}\). For four intervals, the rules are \(\{1\} \not\rightarrow \{3\}, \{1\} \not\rightarrow \{4\}, \{1\} \not\rightarrow \{3, 4\}, \{2\} \not\rightarrow \{4\}\), and \(\{1, 2\} \not\rightarrow \{4\}\).

4.4. Case IV: (Staircase and nonproper intervals). Among three types of overlapping intervals, we have discussed in case II about the type where intervals overlap only with their neighbors. In this subsection, we take a look at the remaining two types of overlapping intervals.
Suppose there are \( n \) (\( n > 2 \)) overlapping intervals \( I_1, I_2, \ldots, I_n \) on \( \mathbb{R} \) where \( I_k = [a_k, b_k] \) for \( 1 \leq k \leq n \) and \( a_1 < a_2 < \ldots < a_n \). Moreover, \( I_1, I_2, \ldots, I_n \) form an \( n \)-staircase. By the definition of a staircase, we have \( a_1 < a_2 < \ldots < a_n < b_2 < \ldots < b_n \). If \( I_k \) covers exactly \( I_i \) through \( I_j \) (\( i \leq j \)) and \( I_{k+2} \) covers exactly \( I_s \) through \( I_t \) (\( s \leq t \)), then the intervals that \( I_{k+1} \) covers must be a subset of \( I_{\min \{i,s\}} \) through \( I_{\max \{j,t\}} \). This is true because if \( I_{k+1} \) covers any interval other than \( I_{\min \{i,s\}} \) through \( I_{\max \{j,t\}} \), then there exists a point \( c \in I_{k+1} \) such that \( f(c) < a_{\min \{i,s\}}-1 \) or \( f(c) > b_{\max \{j,t\}}+1 \). But \( c \in I_k \) or \( c \in I_{k+2} \) by definition of a staircase. Hence, either \( I_k \) or \( I_{k+2} \) would cover \( I_{\min \{i,s\}}-1 \) or \( I_{\max \{j,t\}}+1 \), which is a contradiction.

**Example 4.7.** Let \( I_1 = [a_1, b_1], I_2 = [a_2, b_2], I_3 = [a_3, b_3] \) be three intervals as shown in Figure 4.14. Suppose \( I_1 \) \( f \)-covers \( I_1 \) and \( I_3 \) \( f \)-covers \( I_1 \) and \( I_2 \). As a result, it is impossible that \( I_2 \) \( f \)-covers \( I_3 \).

![Figure 4.14. A 3-staircase](image)

Now we add the possibility that some intervals can be contained in others. Let \( I_j \subset I_k \). It is clear that the intervals \( I_j \) \( f \)-covers is a subset of those \( I_k \) \( f \)-covers. Let \( K \) be the set of intervals \( I_k \) covers and \( J \) be the set of intervals \( I_j \) covers. If none of the elements in \( K \) is contained in another, then all the elements in \( J \) must be a sequence of consecutive intervals and \( J \subset K \).
Example 4.8. Figure 4.15 shows a dynamical system with overlapping intervals $I_1, I_2, I_3$ with $I_3 \subset I_2$ and $I_3 \subset I_1$. In this case, $I_1 \rightarrow I_1 \cup I_2 \cup I_3$, $I_2 \rightarrow I_1 \cup I_2 \cup I_3$, and $I_3 \rightarrow I_1 \cup I_3$. Notice that the two intervals $I_3$ covers is a subset of the intervals that $I_2$ covers and $I_3$ does not cover a sequence of consecutive intervals.
When we discussed different cases of possible and impossible graphs in Section 4, we always assume that the order of intervals is known. Indeed, knowing the interval ordering gives us information about the relationship between the intervals covered by an interval $I_k$ and the intervals covered by $I_{k-1}$ and $I_{k+1}$. However, in many situations, we are just given the intersection graph and the transition graph, where the vertices (intervals) are not labeled. In this section, we will investigate how to determine the order of intervals from a known interval graph. It turns out that interval graphs are a special sub-family of graphs that have many nice properties.

In many applications in mathematical biology, especially in the study of DNA [HSS01, MM99], we often want to know whether a graph is an interval graph or a proper interval graph. Computer scientists and mathematicians have developed algorithms to recognize an interval graph and compute an interval representation for it [KKW12], most of which have linear time complexity. It turns out that, generally speaking, the order of an interval graph’s interval representation is not unique. But the order is unique if we consider proper interval graphs.

**Theorem 5.1** ([DHH96], Corollary 2.5). For each connected proper interval graph $G = (V, E)$, there is a unique ordering (up to full reversal) $v_1, v_2, \ldots, v_n$ of $n$ vertices such that $G$ has a unique proper interval representation $I(G)$ such that $L(I_{v_1}) < L(I_{v_2}) < \ldots < L(I_{v_n})$, where $L(I_{v_i})$ denotes the left endpoint of interval $I_{v_i}$.

In [DHH96], Deng, Hell, and Huang provide an elegant algorithm (the DHH algorithm) to generate representations of proper interval graphs, which has time complexity $O(m + n)$, where $m$ and $n$ are number of of vertices and edges of the given proper interval
graph. It does not require constructing a complicated data structure, such as a PQ-tree (see [BL76] for details). Before further explaining the DHH algorithm, we need to define some terms.

**Definition 5.2.** The claw, the net, and the tent are graphs isomorphic to those in Figure 5.16.

![The claw, the net, and the tent](image)

**Figure 5.16.** The claw, the net, and the tent

**Definition 5.3.** Let $G = (V, E)$ be a graph. For each vertex $v \in V$ we define $N(v) = \{u \in V : (u, v) \in E\}$ and $N[v] = N(v) \cup \{v\}$. Let $R$ be the equivalence relation on $V$ defined by $uRv$ if and only if $N[u] \equiv N[v]$. Each equivalence class of $R$ is called a *block* of $G$.

**Definition 5.4.** An *orientation* of an undirected graph is an assignment of a direction to each edge. An *oriented graph* is a digraph that is an orientation of some undirected graph.

**Definition 5.5.** A *tournament* is a digraph obtained by assigning a direction to each edge in an undirected complete graph. A *transitive tournament* is a tournament $T = (V, A)$ such that for any $a, b, c \in V$, if $(a, b) \in A$ and $(b, c) \in A$, then $(a, c) \in A$. 

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Definition 5.6. Let $D = (V, A)$ be a digraph. For each $v_i \in V$, $v_k$ is in the *inset* of $v_i$ if and only if $(v_k, v_i) \in A$ and $v_l$ is in the *outset* of $v_i$ if and only if $(v_i, v_l) \in A$. The cardinality of the outset of $v_i$ is called the *outdegree* of $v_i$.

Definition 5.7. A *straight enumeration* of an oriented graph $D$ is a linear ordering $v_1, v_2, \ldots, v_n$ of its vertices such that for each $i$ there exist nonnegative integers $k$ and $l$ such that the vertex $v_i$ has inset $\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}$ and outset $\{v_{i+1}, v_{i+2}, \ldots, v_{i+l}\}$.

Definition 5.8. An oriented graph which admits a straight enumeration is called *straight*. An undirected graph has a *straight orientation* if it admits an orientation which is a straight oriented graph.

Algorithm 5.9 (The DHH algorithm). Let $G = (V, E)$ be a connected graph.

[Step 1] Order the vertices of $G$ as $v_1, v_2, \ldots, v_n$ in such a way that the subgraph induced by $\{v_1, v_2, \ldots, v_i\}$ is connected for each $i = 1, 2, \ldots, n$. Let $v = v_1$ and $H = \{v\}$. 

[Step 2] If $H = \{v_1, v_2, \ldots, v_i\}$, then let $v = v_{i+1}$ and insert $v$ into $H$. Let $G_H$ be the subgraph of $G$ induced by $H$. Repeat this step as long as $G_H$ is chordal and does not contain the claw, the net, or the tent as an induced subgraph.

[Step 3] If $H$ does not contain all vertices of $G$, then report that $G$ is not a proper interval graph. Otherwise proceed to Step 4.

[Step 4] Orient $G = (V, E)$ into $D = (U, A)$ such that each block is a transitive tournament and $u_1, u_2, \ldots, u_n$ is a straight enumeration of $D$.

[Step 5] Associate to each vertex $u_i$ the interval $[i, i + d_i + 1 - \frac{1}{i}]$, where $d_i$ is the outdegree of $u_i$. 

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The purpose of the DHH algorithm is to first determine if the given graph is a proper interval graph and then give an interval representation the graph passes the test. We illustrate the algorithm with some examples.

**Example 5.10.** Let $G$ be the graph in Figure 5.17. Earlier we used Theorems 2.8 and 2.10 to conclude that this graph is not a proper interval graph. We will now use the DHH algorithm to reach the same conclusion.

![Graph](image)

**FIGURE 5.17.** An example of a nonproper interval graph

Notice that $G$ is connected and $v_1, v_2, \ldots, v_7$ is an order of the vertices such that the subgraph induced by $\{v_1, v_2, \ldots, v_i\}$ is connected for each $i = 1, 2, \ldots, 7$.

Let $H = \{v_1\}$. The subgraph induced by $H$ is the first graph in Figure 5.18, which clearly does not contain a claw, a net, or a tent as an induced subgraph.

Now we add $v_2$ to $H$. Thus, $H = \{v_1, v_2\}$. Since the subgraph induced by the new $H$ (the second in Figure 5.18) does not contain a claw, a net, or a tent as an induced subgraph, we can keep adding vertices in the order to $H$. By step 2 of the DHH algorithm, we can keep adding vertices to $H$ until $H = \{v_1, v_2, \ldots, v_5\}$.

If we add $v_6$ to $H$, the induced subgraph of $H$ will be the sixth in Figure 5.18. However, the subgraph induced by $\{v_1, v_3, v_5, v_6\}$ of the sixth graph in Figure 5.18 is a claw. Therefore, $G$ is not a proper interval graph.
It is worth noticing that the order of the vertices we chose at step 1 is arbitrary, as long as the subgraph induced by the first $i$ vertices in the order is connected for $i = 1, 2, \ldots, 7$. Figure 5.19 gives two possible nonproper interval representations of $G$. In the first graph, $I_5 \subset I_3$, while in the second graph, $I_5 \subset I_4$. Notice the order of the left points of the intervals is not unique.

The graph in Example 5.10 is not a proper interval graph. Hence we cannot apply the algorithm to transform it into a proper interval representation. Now let’s look at a graph.
that can pass the proper interval graph test and go through the process of developing a proper interval representation.

**Example 5.11.** Let $G$ be the graph in Figure 5.20. Similarly, we conclude that it is a proper interval graph using Theorems 2.8 and 2.10. We obtain the same conclusion using the DHH algorithm.

![Figure 5.20. A proper interval graph](image)

Notice that $v_1, v_2, \ldots, v_7$ is an order of the vertices such that the subgraphs induced by \{v_1, v_2, \ldots, v_i\} are connected for $i = 1, 2, \ldots, 7$. Figure 5.21 shows its induced subgraphs in this particular order. Clearly, none of the induced subgraphs in the figure contain a claw, a net, or a tent as an induced subgraph. Hence, we conclude that $G$ is a proper interval graph.

After identifying the blocks, we must assign a direction to the edges in each subgraph induced by each block such that they become transitive tournaments. There is nothing to do for the subgraphs induced by the blocks containing only one vertex, since the induced subgraphs do not contain any edges. Consider the block \( \{v_3, v_4\} \). Because there is only one edge in its induced subgraph, it does not matter if we assign the direction to be \((v_3, v_4)\) or \((v_4, v_3)\). Without loss of generality, we choose \((v_4, v_3)\) to be the direction of the edge as shown in Figure 5.22.
Next, we must find a straight orientation of the graph $G$. This process sometimes requires renaming the vertices. In this case, we rename $v_1, v_2, v_5, v_6, v_7$ to $u_1, u_2, u_5, u_6, u_7$ respectively, and rename $v_3$ to $u_4$ and $v_4$ to $u_3$. Notice that if we chose the direction of the edge between $v_3$ and $v_4$ to be $(v_3, v_4)$, we would not need to rename the vertices. A straight orientation of $G$ is shown in Figure 5.23. We call this new directed graph $D = (U, A)$.

![Diagram of the graph $D$](image)

**Figure 5.23. A straight orientation of $G$**

Finally, we associate to each vertex $u_i$ the interval $[i, i + d_i + 1 - \frac{1}{i}]$, where $d_i$ is the outdegree of $u_i$. Hence,

$I_{u_1} = [1, 1 + 1 + 1 - \frac{1}{1}] = [1, 2]$

$I_{u_2} = [2, 2 + 2 + 1 - \frac{1}{2}] = [2, 4\frac{1}{2}]$

$I_{u_3} = [3, 3 + 2 + 1 - \frac{1}{3}] = [3, 5\frac{2}{3}]$

$I_{u_4} = [4, 4 + 1 + 1 - \frac{1}{4}] = [4, 5\frac{3}{4}]$

$I_{u_5} = [5, 5 + 1 + 1 - \frac{1}{5}] = [5, 6\frac{4}{5}]$

$I_{u_6} = [6, 6 + 1 + 1 - \frac{1}{6}] = [6, 7\frac{5}{6}]$

$I_{u_7} = [7, 7 + 0 + 1 - \frac{1}{7}] = [7, 7\frac{6}{7}]$.  

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Figure 5.24 shows what the intervals look like on the real number line.

![Interval Representation](image)

**Figure 5.24. A proper interval representation given by the DHH algorithm**

It is interesting to notice that the enumeration of vertices within each block does not alter the proper interval representation that the DHH algorithm generates. Moreover, if we take different enumerations of vertices within the block and come up with straight enumerations of all vertices in the graph, we would essentially get the same straight orientation. In Example 5.11, if we chose the direction of the edge in the block to be \((v_3, v_4)\) instead of \((v_4, v_3)\), the positions of vertices \(u_3, u_4\) would change at the step of finding a straight orientation of \(G\). However, the two oriented graphs of \(G\) coming from different edge direction assignments within the block are isomorphic. Therefore, the edge direction assignment within each block is independent of the proper interval representation provided by the algorithm or the interval order of the proper interval representation.

Although the order of vertices in each block does not matter when we look for a proper interval representation, it does make a difference when we try to find a realization of a dynamical system, considering the transition graph and the intersection graph at the same time.
Example 5.12. In the last example, we observed that vertices $u_3$ and $u_4$ are interchangeable. But we need to be more careful when we are given a transition graph as well. In Figure 5.25, the intersection graph is the same as that in Example 5.11. However, $I_{u_3}$ must be the interval representation of $v_3$ and $I_{u_4}$ must be the interval representation of $v_4$. Because if $I_{u_3}$ is the interval representation of $v_4$, then $I_{u_3}$ covers $I_{u_3}$ and $I_{u_5}$ according to the transition graph, which is certainly impossible as discussed in Section 4.1. A solution to this problem is to consider every ordering in each vertex block and check its compatibility with the transition graph. (By compatible, we mean that such ordering makes the transition graph a possible one.) In more complicated graphs, there may be more than one ordering within a block that is compatible with the transition graph, which adds more possibilities of function realization.

Figure 5.26 is a realization of Figure 5.25.

Since we have the ability to find proper interval representations, we can use the conclusions in Section 4 to determine whether a given transition graph is possible (after
FIGURE 5.26. A dynamical system realization of the graph in Figure 5.25 considering all orderings within each block) if the structure of the proper interval representation falls into one of the categories in Section 4. But if the proper interval representation consists of multiple structures mentioned in Section 4, we currently do not have a rule of thumb to determine if the transition graph is possible or not.

In the next section, we will say more about how to find functions that realize a given pair of transition graph and intersection graph. We will show techniques to construct realizations in the form of piecewise linear functions.
6. Construction of piecewise linear functions

If we are given an intersection graph and a transition graph (the outset of every vertex is nonempty) and we know that it can be realized on the real number line, we can always find a piecewise linear function realization. The construction process is not very difficult. Let \( I_k = [a_k, b_k] \) be intervals with the given intersection graph. For example, if the interval graph is proper, this could be the result of the DHH algorithm.

Suppose the intersection graph does not have any edges. Then the intervals are disjoint. We may assume that the order of the intervals on \( \mathbb{R} \) is \( I_1, I_2, \ldots, I_n \). Then, if \( I_i \) \( f \)-covers \( I_j, I_{j+1}, \ldots, I_k \). We can map \( a_i \) to \( a_j \) and \( b_i \) to \( b_k \). Then connect \((a_i, a_j)\) and \((b_i, b_k)\) with a straight line. Last, we connect the points at \( b_i \) and \( a_{i+1} \) for \( 1 \leq i \leq n - 1 \) with straight lines as well.

**Example 6.1.** Figure 6.27 shows an intersection graph, transition graph pair. And the intersection graph indicates that the intervals are disjoint. According to the transition graph, \( I_1 \) \( f \)-covers \( I_2 \) and \( I_3 \). So we map \( I_1 \) on the horizontal axis to a straight line across \( I_2 \) and \( I_3 \) on the vertical axis. Then use the same strategy to draw \( I_2 \rightarrow I_3 \) and \( I_3 \rightarrow I_1 \). In the end, we connect the pieces by straight lines.

![Figure 6.27. Construct piecewise linear functions on disjoint intervals](image-url)
If the intervals are given in order and they only overlap with their neighbors, then there exists \([c_i, d_i] \subset I_i\) such that \([c_i, d_i] \cap I_{i-1} = \emptyset\) and \([c_i, d_i] \cap I_{i+1} = \emptyset\) for each \(i\). Notice that intervals \([c_i, d_i]\) are disjoint. We can mimic the procedure in the disjoint interval case, but without connecting between the intervals. Next, we only connect two pieces by a straight line if it would not make either interval cover more intervals than what the transition graph indicates. If connecting \([c_i, d_i]\) and \([c_{i+1}, d_{i+1}]\) violates the transition graph, then find points \(d'_i\) and \(c'_{i+1}\) such that \(d_i < d'_i < a_{i+1}\) and \(b_i < c'_{i+1} < c_{i+1}\).

Do either or both of the following (whichever does not violate the transition graph after connecting the lines in \(I_i\) and \(I_{i+1}\)):

i) Connect \((d_i, f(d_i))\) and \((d'_i, f(c_i))\).

ii) Connect \((c'_{i+1}, f(d_{i+1}))\) and \((c_{i+1}, f(c_{i+1}))\).

**Example 6.2.** Let \(I_1, I_2, \ldots, I_5\) be intervals shown as in Figure 6.28. Suppose the transition graph indicates that \(I_1 \rightarrow I_4 \cup I_5\) and \(I_2 \rightarrow I_1 \cup I_2\).

![Figure 6.28. Construct piecewise linear functions on intervals overlap with neighbors](image)
We find two subsets of $I_1$ and $I_2$ respectively as described and follow the first step of the procedure in the disjoint interval case. As shown in the graph on the left, if we simply connect the two pieces, $I_1$ would cover $I_2, I_3, I_4$ and $I_5$, instead of just $I_4$ and $I_5$. In this case, we need to perform both i) and ii) to somehow "flip" the linear piece so that when we connect them afterwards, it does not violate the transition graph. Hence, we obtain the piecewise linear function in the graph on the right.

For other intersection graphs, we partition the intervals into smaller pieces sharing endpoints, where each $a_i$ or $b_i$ is an endpoint. (But for nonproper interval graphs, since their interval representations are not unique, we want to perform the procedure to every possible interval realization.) For example, we can break the intervals in Figure 5.24 into 13 small intervals shown in Figure 6.29.

Next, we construct a new transition graph based on the given transition graph. Again, using the same example, the original transition graph in Figure 5.25 indicates that $I_{u_1} \rightarrow I_{u_2}$ and $I_{u_2} \rightarrow I_{u_3} \cup I_{u_4}$. Since $J_1 \subset I_{u_1}$ and $J_1$ is not a subset of any other original intervals, we can write $J_1 \rightarrow I_{u_2}$, or equivalently, $J_1 \rightarrow J_2 \cup J_3 \cup J_4 \cup J_5$. Additionally, $J_2 = I_{u_1} \cap I_{u_2}$. Thus, we say that $J_2$ covers $(I_{u_2}) \cap (I_{u_3} \cup I_{u_4})$, or equivalently, $J_2 \rightarrow J_4 \cup J_5$. 
Similarly, we can write out what the other 11 intervals $f$-cover. Now, we have transformed it into the case where our new intervals only overlap with their neighbors. Therefore, following the previous procedure, we obtain Figure 6.30.

**Figure 6.30.** A piecewise linear function for Figure 5.25
7. Classifying possible and impossible graphs of one-dimensional dynamical systems on a circle

Besides the real number line, a circle is the only other one-dimensional manifold. We can study continuous dynamical systems on a circle like we did for the real number line. We can also study its symbolic dynamics, when intervals are constructed along the circle as shown in Figure 7.31.

![Intervals on a circle](image)

**Figure 7.31.** Intervals on a circle

On the real number line, we enumerate intervals by their left endpoints. By a similar manner, we can enumerate intervals on a circle in counter-clockwise fashion. For a continuous function on a real number line, each interval must $f$-cover a consecutive sequence of intervals. This is also true for the circle. Notice that in Figure 7.31, $I_4$ is adjacent to $I_1$. So an interval can $f$-cover only $I_1$ and $I_4$.

7.1. **Case I: (Disjoint intervals).** If the intervals are disjoint, we can calculate the number of possible transition graphs on a circle with $n$ intervals. The number will be much bigger than what we computed in Section 4.1, because there are more combinations of consecutive intervals on a circle. Suppose there are $n$ disjoint intervals on a circle (Figure 7.32).
For each interval, under a continuous function $f$, there are $n$ ways to cover exactly one interval, $n$ ways to cover exactly two intervals, \ldots, $n$ ways to cover exactly $n-1$ intervals, and one way to cover all $n$ intervals. Therefore, there are $n(n-1)+1$ possible combinations of outgoing edges for each vertex (interval) and there are $n$ vertices (intervals). So the number of possible transition graphs with $n$ vertices is $(n(n-1)+1)^n$, much larger than that in the real number line case \( \left(\frac{n(n+1)}{2}\right)^n \).

7.2. Case II: (Intervals overlapping only with their neighbors). Suppose there are $n$ intervals on a circle and each interval only overlaps with its neighbors. Recall that in the real number line case (Section 4.2), if $I_k \cap I_{k+1} \neq \emptyset$, $f(I_k) \supset I_i \cup I_{i+1} \cup \ldots \cup I_j$, and $f(I_{k+1}) \supset I_s \cup I_{s+1} \cup \ldots \cup I_t$, then either $j+2 \geq s$ or $t+2 \geq i$. We cannot directly use the previous result since the subscripts of intervals that $I_k$ covers may not be consecutive integers. However, the analogous result holds in this cyclic ordering. The two sets of consecutive intervals that $I_k$ and $I_{k+1}$ $f$-cover respectively cannot be more than two intervals apart on the circle.

7.3. Case III: (Intervals sharing endpoints that map to other endpoints). This is a special case of case II. Suppose $f$ is a continuous function on the circle and there are $n$
Figure 7.33. $n$ intervals sharing endpoints on a circle

intervals $I_1, I_2, \ldots, I_n$ on a circle with $I_i = [a_{i-1}, a_i]$ and $a_0 = a_n$ (Figure 7.33). Moreover $f(a_i) = a_j$ for some $j$. That is, all endpoints are eventually periodic. For any interval $I_k = [a_{k-1}, a_k]$ with $f(a_{k-1}) = a_i$ and $f(a_k) = a_j$, the intervals that $I_k$ covers must contain all intervals on either side of the secant line from $a_i$ to $a_j$. For example, if $f(a_{k-1}) = a_1$ and $f(a_k) = a_{n-2}$, then either $f(I_k) \supset \{I_1, I_n, I_{n-1}\}$ or $f(I_k) \supset \{I_2, I_3, \ldots, I_{n-2}\}$ as shown in Figure 7.34.

Figure 7.34. An example of $f(a_{k-1}) = a_1$ and $f(a_k) = a_{n-2}$. 

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7.4. **Interval ordering on a circle.** In graph theory, the intersection graph of intervals on a circle is called a circular-arc graph. In [DHH96], the same paper where the clever algorithm of finding proper interval representation was introduced, Deng, Hell, and Huang also provided an algorithm recognizing proper circular-arc graphs and finding a circular-arc representation. Additionally, they proved that certain types of circular-arc graphs have unique circular-arc representations up to full reversal.

The algorithm that recognizes a proper circular-arc graph and finds a circular-arc representation utilizes Tucker’s algorithm [Tuc71] and Algorithm 5.9. Tucker’s algorithm has $O(n^2)$ time complexity. According to Deng, Hell, and Huang, Tucker’s algorithm is linear when $m \geq \frac{n^2 - 2n}{4}$, where $m$ is the number of edges and $n$ is the number of vertices [DHH96]. The algorithm for circular-arc graphs with $O(n + m)$ time complexity that Deng, Hell, and Huang developed is much more complicated than Algorithm 5.9. We will not discuss it in detail. The general idea of the algorithm is to check if the graph is a proper interval graph first using Algorithm 5.9. If it is, then treat it as a special case of circular arcs: simply wrapping up an interval graph. If it is not, then break it into two cases: one uses Tucker’s algorithm and the other further breaks the graph into pieces and applies Algorithm 5.9 to some pieces of the graph to determine whether the graph is a proper circular-arc graph or not.

**Definition 7.1.** A graph $G$ is *bipartite* if and only if all vertices of $G$ can be divided into two disjoint sets $U$ and $V$ such that each edge in $G$ connects a vertex in $U$ and a vertex in $V$.

Notice that $K_{1,3}$, which we discussed in Section 2, is a bipartite graph.
Definition 7.2. A graph $G$ is a *mixed graph* if some of its edges are directed and some of them are undirected.

Definition 7.3. A graph $G = (V, E)$ is a *round graph* if the vertices in $G$ can be circular enumerated and for each $v \in V$, either $N(v)$ or $N[v]$ forms an interval in the enumeration.

Definition 7.4. The *complement* of a graph $G = (V, E)$ is defined as $\bar{G} = (V, \bar{E})$ such that $\bar{E} = \{ \{x, y\} \notin E | x, y \in V \}$.

Theorem 7.5 ([DHH96]). Let $G$ be a connected proper circular-arc graph. If $G$ is connected or nonbipartite, then it is uniquely orientable as a round mixed graph up to full reversal.

Notice that the unique orientation is not true for any proper circular-arc graphs, which is the major difference between the uniqueness of a proper circular-arc representation and a proper interval representation. Nevertheless, if the proper circular-arc graph satisfies all the requirements stated in Theorem 7.5, we can be certain that the orientation of the circular-arc representation we found using the DHH algorithm is unique. Similarly, if the circular-arc representation has a structure that falls into any of the categories mentioned in the beginning of Section 6, we can apply the rules to check if the transition graph is possible with considering every ordering in each block.
8. CONCLUSION AND FUTURE WORK

We have discussed 1) how to obtain the intersection graph and transition graph from a given set of intervals and a function, and 2) how to find a realization of a given pair of intersection graph and transition graph. While the first part helps us understand the nature of the superimposed graph and what kind of information it carries, the second part is more useful when we want to construct examples of one-dimensional symbolic dynamical systems with overlapping regions having specific properties.

In the future, we can further study how to construct realizations on $S^1$. Moreover, we can expand the analysis to two-dimensional dynamical systems with overlapping regions. It would be much more challenging. In one-dimensional spaces, there are only two ways of overlapping: complete containment and partial overlapping. However, there are more types of overlapping in two-dimensional spaces.
REFERENCES


