4-2003

Bounded Homeomorphisms of the Open Annulus

David S. Richeson
Dickinson College

Jim Wiseman

New York Journal of Mathematics

New York J. Math. **9** (2003) 55–68.

Bounded homeomorphisms of the open annulus

David Richeson and **Jim Wiseman**

Abstract. We prove a generalization of the Poincaré-Birkhoff theorem for the open annulus showing that if a homeomorphism satisfies a certain twist condition and the nonwandering set is connected, then there is a fixed point. Our main focus is the study of bounded homeomorphisms of the open annulus. We prove a fixed point theorem for bounded homeomorphisms and study the special case of those homeomorphisms possessing at most one fixed point. Lastly we use the existence of rational rotation numbers to prove the existence of periodic orbits.

Contents

1. Introduction

A homeomorphism $f : X \to X$ is said to be *bounded* if there is a compact set which intersects the forward orbit of every point. Since every homeomorphism on a compact space is bounded, bounded homeomorphisms are interesting only on noncompact spaces. If f is bounded then there is a forward invariant compact set which intersects the forward orbit of every point (see Theorem 2). Thus, a bounded map on a noncompact space behaves in many ways like a map on a compact space. In particular, many results that are true for maps on compact spaces are also true for bounded maps on noncompact spaces (e.g., the Lefschetz fixed point theorem).

In this paper we study primarily the dynamics of bounded homeomorphisms of the open annulus. Intuitively we may view these homeomorphisms as those having repelling boundary circles. In fact, we will see that the orbit of every point

Received December 19, 2002.

Mathematics Subject Classification. Primary 37E40; Secondary 37E45, 54H25.

Key words and phrases. Annulus, Poincaré-Birkhoff theorem, twist map, fixed point, nonwandering set, periodic point, rotation number.

The second author was partially supported by the Swarthmore College Research Fund.

ISSN 1076-9803/03

intersects an essential, closed, forward invariant annulus. Thus, roughly speaking, many of the results for homeomorphisms of the closed annulus also hold for bounded homeomorphisms of the open annulus. Conversely, many of the results that hold for bounded homeomorphisms of the open annulus also hold for homeomorphisms of the closed annulus; one may enlarge the closed annulus to an open annulus and extend the homeomorphism to a bounded homeomorphism of this open annulus.

The most celebrated result for the closed annulus is the Poincaré-Birkhoff theorem (also called Poincaré's last geometric theorem), which states than any area preserving homeomorphism which twists the boundary components in opposite directions has at least two fixed points. In [Fra88a] Franks gives a topological generalization for the open annulus; he proves that if every point in an open annulus is nonwandering and f satisfies a twist condition, then there is a fixed point of positive index. We prove a further generalization showing that if f satisfies a twist condition and the nonwandering set is connected then f has a fixed point. Recall that for a map $f : X \to X$, a point $x \in X$ is *nonwandering* if for every open set U containing x there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The collection of nonwandering points is the *nonwandering set*, denoted $\Omega(f)$.

The paper is divided as follows. In Section 2 we present general properties of bounded homeomorphisms of the annulus. In Section 3 we prove a generalization of the Poincaré-Birkhoff-Franks theorem for the open annulus. This section applies to homeomorphisms of the open annulus that need not be bounded. It can be read independently of the rest of the paper and may be of more general interest. In Section 4 we use this theorem to prove a fixed point theorem for bounded homeomorphisms of the open annulus. It is interesting to note that a bounded homeomorphism of a noncompact space can never preserve Lebesgue measure (see Corollary 3). Thus, we prove a fixed point theorem for a family of maps far from satisfying the hypotheses of the Poincaré-Birkhoff theorem. Also, we study the special case of those bounded homeomorphisms having at most one fixed point. Lastly, in Section 5 we apply the theorem to those bounded homeomorphisms having a point with a rational rotation number and prove the existence of a periodic point with that same rotation number.

In this paper we will let A denote the annulus $(\mathbb{R}/\mathbb{Z}) \times I$, where $I = [0, 1]$ if A is the closed annulus, and $I = (0, 1)$ if A is the open annulus. $\widetilde{A} = \mathbb{R} \times I$ will denote the universal cover of the annulus A with $\pi : \widetilde{A} \to A$ being the covering projection. We view $\widetilde{A}$ as a subset of $\mathbb{R}^2$, thus when we subtract two elements in $\widetilde{A}$ we obtain a vector in $\mathbb{R}^2$. The projection onto the first coordinate $\mathbb{R}^2 \to \mathbb{R}$ is given by $(x, y)_1 = x$. For any set $U \subset \widetilde{A}$, let $U + k$ denote the set $\{(x+k, y) \in \widetilde{A} : (x, y) \in U\}$.

If $f : A \to A$ is a homeomorphism then there is a *lift*, $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ satisfying $\pi \circ \widetilde{f} = f \circ \pi$. Notice that $\widetilde{g}$ is another lift of f iff $\widetilde{g}(x, y) = \widetilde{f}(x, y) + (k, 0)$ for some integer k. For any $y \in \widetilde{A}$ define $\rho(y, \widetilde{f})$ to be $\lim_{n\to\infty} (1/n)(\widetilde{f}^n(y) - y)_1$ (if this limit exists). If $\widetilde{g}$ is another lift then $\rho(y, \widetilde{g}) = \rho(y, \widetilde{f}) + k$ for some integer k. Thus we may define the *rotation number* of $x = \pi(y) \in A$ to be $\rho(x) = \rho(y, \widetilde{f}) \pmod 1$ if this limit exists. So defined, $\rho(x)$ is independent of the choice of y and $\widetilde{f}$. Unlike the case of homeomorphisms of the circle, for homeomorphisms of the annulus different points may have different rotation numbers, and it may happen that the rotation number for a point does not exist.

2. Bounded homeomorphisms of the annulus

ln [RW02] the authors introduced the following definitions.

Definition 1. A compact set W is a *window* for a dynamical system on X if the forward orbit of every point $x \in X$ intersects W. If a dynamical system has a window then we will say that it is *bounded.*

We showed that we can characterize bounded dynamical systems in many ways. The following theorem summarizes some results from [RW02].

Theorem 2. *Let $f : X \to X$ be a continuous map on a locally compact space X. Then the following are equivalent:*

1. *f is bounded.*
2. *There is a forward invariant window.*
3. *Given any compact set $S \subset X$ there is a window $W \subset X$ containing S such that $f(W) \subset \operatorname{Int} W$.*
4. *There is a compact set $W \subset X$ with the property that $\omega(x)$, the omega-limit set of x, is nonempty and contained in W for all $x \in X$.*
5. *f has a compact global attractor Λ (i.e., Λ is an attractor with the property that for every $x \in X$, $\omega(x)$ is nonempty and contained in Λ).*

Because every bounded map has a compact global attractor it is impossible for it to preserve Lebesgue measure on a noncompact space. Thus we have the following corollary.

Corollary 3. *Suppose $f : X \to X$ is an area preserving map of a noncompact space X. Then f is not bounded. In particular, if $S \subset X$ is any compact set, then there exists a point $x \in X$ such that the forward orbit of x does not intersect S.*

Example 4. Consider a convex billiards table (for an introduction to billiards and billiard maps see [KH95]). Is it possible to find a trajectory with the property that the angle the ball makes with the wall is always smaller than some arbitrarily chosen ε? We see that the answer is yes.

Let $f : S^1 \times (0, \pi) \to S^1 \times (0, \pi)$ be the billiards map corresponding to the given table. It is well-known that f is an area preserving homeomorphism homotopic to the identity. By Corollary 3, f is not bounded. In particular, there exists a point (x, θ) whose forward orbit does not intersect the closed annulus $S^1 \times [\varepsilon, \pi - \varepsilon]$.

Thus, for any $\varepsilon > 0$, there exists a trajectory $(x_0, \theta_0), (x_1, \theta_1), (x_2, \theta_2), \dots$ such that either $\theta_k < \varepsilon$ for all $k \geq 0$ or $\pi - \theta_k < \varepsilon$ for all $k \geq 0$.

Example 5. Suppose there is a convex billiards table with bumpers in the middle of the table (see Figure 1). Is it possible to find a trajectory of the billiards ball that never strikes a bumper?

Assume that the bumpers are a finite collection of compact sets not touching the wall of the billiards table. Consider the billiards map for the table with no bumpers, $f : S^1 \times (0, \pi) \to S^1 \times (0, \pi)$. Let $W \subset S^1 \times (0, \pi)$ be the set of points $\{(x, \theta)\}$ with the property that a ball at position x with trajectory angle θ will strike a bumper before striking the wall again. Clearly W is a compact set. Thus, we rephrase the question: Is it possible to find an orbit of f that never intersects W? By the discussion in Example 4 it is clear that such a trajectory does exist. Thus, given any compact set of bumpers, there is always a trajectory that avoids the bumpers.

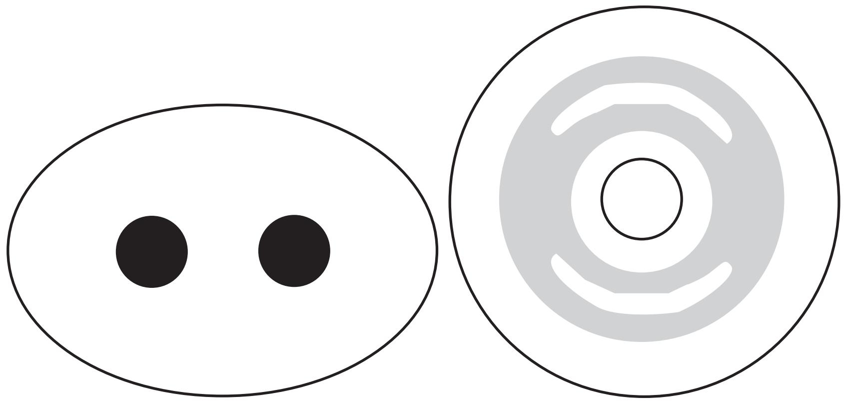

FIGURE 1. The billiards table with two bumpers and the corresponding configuration space

The following proposition follows from more general results in [Gün95, §4], but under our hypotheses we can give a shorter, more dynamical proof.

Proposition 6. *Suppose $f : A \to A$ is a bounded homeomorphism of the open annulus with a compact global attractor $\Lambda \subset A$. Then the following are true:*

1. *The inclusion $i : \Lambda \to A$ induces an isomorphism on Čech cohomology, $i^* : \check{H}^*(A) \to \check{H}^*(\Lambda)$.*
2. *Λ is connected.*
3. *Λ separates the two boundaries of A.*

Proof. Let $f : A \to A$ be a bounded homeomorphism of the open annulus A with compact global attractor Λ. By Theorem 2 there exists a window W such that $\Lambda \subset f(W) \subset \operatorname{Int} W$. Let $\varepsilon > 0$ be small enough such that $\Lambda \subset A_\varepsilon = [\varepsilon, 1-\varepsilon]$. For each $x \in A_\varepsilon$ there exists $n_x > 0$ such that $f^{n_x}(x) \subset \operatorname{Int} W$. There exists an open set U_x containing x such that $f^{n_x}(U_x) \subset \operatorname{Int} W$. The collection $\{U_x\}$ is an open cover of A_ε, thus there exists a finite subcover, $\{U_{x_1}, \ldots, U_{x_m}\}$. Let $N = \max\{n_{x_1}, \ldots, n_{x_m}\}$. It follows that $f^i(A_\varepsilon) \subset \operatorname{Int} W$ for all $i \geq N$.

Notice that $f^N(A_\varepsilon)$ separates the two boundaries of A and f^N induces an isomorphism on cohomology. Also, $U = \operatorname{Int}(A_\varepsilon), f^N(U), f^{2N}(U), \ldots$ is a nested sequence of open sets with $\Lambda = \bigcap_{k=0}^{\infty} f^{kN}(U)$. Consequently, the inclusion $i : \Lambda \to A$ induces an isomorphism $i^* : \check{H}^*(A) \to \check{H}^*(\Lambda)$ and Λ separates the two boundaries of A. Moreover, since Λ is the intersection of a nested collection of connected open sets, Λ is itself connected. □

Next we prove a key result that states that all of the interesting dynamics occurs inside a closed annulus. This result is very useful. It validates our intuition that a bounded homeomorphism on the open annulus behaves like a homeomorphism on the closed annulus.

Proposition 7. *If $f : A \to A$ is a bounded homeomorphism of an open annulus, then there exists a closed annulus $A_0 \subset A$ whose boundaries are smooth essential curves such that $f(A_0) \subset \operatorname{Int} A_0$. Moreover, A_0 can be chosen so that the boundary is as close to Λ or as close to the boundary of A as desired.*

Proof. Let $f : A \to A$ be a bounded homeomorphism of the open annulus $A = S^1 \times (0,1)$. By Theorem 2 there exists a compact global attractor $\Lambda \subset A$. Let $\varepsilon > 0$ (ε should be small enough that $\Lambda \subset S^1 \times [\varepsilon, 1-\varepsilon]$). We will construct a closed annulus A_0 satisfying the conclusion of the theorem with the property that $[\varepsilon, 1-\varepsilon] \subset A_0$. A similar argument can be used to show that we can find A_0 with the boundary close to Λ.

Let $A^* = A \cup \{*\}$ be the one point compactification of A. It is easy to see that $(\Lambda, \{*\})$ is an attractor-repeller pair (in the sense of Conley [Con78]). Let $\gamma : A^* \to \mathbb{R}$ be a continuous Lyapunov function satisfying $\gamma^{-1}(0) = \Lambda$, $\gamma^{-1}(1) = \{*\}$ and $\gamma(f(x)) < \gamma(x)$ for all $x \notin (\Lambda \cup \{*\})$ (see [Fra82] for details). For the remainder of the proof we will restrict γ to be a function from A to $\mathbb{R}$. Let $c \in (0,1)$ be such that $\gamma^{-1}(c) \cap (S^1 \times [\varepsilon/2, 1-\varepsilon/2]) = \emptyset$. Because γ may not be smooth the set $\gamma^{-1}(c)$ could be quite complicated. For any smooth function $\lambda : A \to \mathbb{R}$ (which may not be a Lyapunov function) sufficiently C^0-close to γ and any regular value for λ, $c' \in \mathbb{R}$, sufficiently close to c, $\lambda^{-1}(c') \cap (S^1 \times [\varepsilon, 1-\varepsilon]) = \emptyset$ and $\lambda^{-1}(c') \cap f(\lambda^{-1}(c')) = \emptyset$. Because c' is a regular value, $\lambda^{-1}(c')$ is the disjoint union of smoothly embedded circles in A. By Proposition 6, Λ separates the two boundaries of A. Thus there is at least one circle in $\lambda^{-1}(c')$ that separates Λ from the inside boundary and another circle that separates Λ from the outside boundary. The region bounded by these two circles is a closed annulus A_0 with $[\varepsilon, 1-\varepsilon] \subset A_0 \subset A$ and $f(A_0) \subset \operatorname{Int} A_0$. $\square$

Corollary 8. *If $f : A \to A$ is a bounded homeomorphism of the open annulus homotopic to the identity, then the Lefschetz index of the fixed point set is zero. In particular, if f has a fixed point of nonzero index, then f has at least two fixed points.*

Proof. Suppose $f : A \to A$ is bounded. Then there exists an essential closed annulus $A_0 \subset A$ containing the fixed point set with the property that $f(A_0) \subset \operatorname{Int} A_0$. So, the fixed point set of f has Lefschetz index zero. Clearly, if f has a fixed point of nonzero index, then f has at least two fixed points. $\square$

3. A generalization of the Poincaré-Birkhoff theorem

The classical Poincaré-Birkhoff Theorem states that every area preserving homeomorphism of the closed annulus that twists the two boundary components in opposite directions must have two fixed points ([Poi12], [Bir25], [Bir13]). In the years since it was proved there have been new proofs and various generalizations (see for instance [BN77], [Fra88a], [Fra88c], [Car82], [Gui97], [Win88], [AS76]). In [Fra88a] Franks generalizes this theorem to the open annulus. He weakens the area preserving hypothesis to the assumption that every point is nonwandering and he weakens the twist condition to one about positively and negatively returning disks. The expense of these assumptions is that the homeomorphism may have only one fixed point, but this fixed point has positive index.

In this section we observe that we may weaken the hypotheses to the assumption that the nonwandering set, $\Omega(f)$, is connected. In this case the homeomorphism must have a fixed point (now possibly of zero index). Since there are now points that are not nonwandering we must clarify the twist condition - we will insist that the positively and negatively returning disks intersect the nonwandering set.

Definition 9. Let $f : A \to A$ be a homeomorphism of an open or closed annulus and let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be a lift of f. An open disk $U \subset \widetilde{A}$ is a *positively returning disk* if $\widetilde{f}(U) \cap U = \emptyset$, if $\pi(U)$ is a disk in A, and if there exist $n, k > 0$ such that $\widetilde{f}^n(U) \cap (U + k) \neq \emptyset$. Define the set $\Omega^+(f) = \{y \in \Omega(f) : y \in \pi(U)$ for some positively returning disk $U\}$. Similarly, define *negatively returning disks* (requiring $k < 0$) and $\Omega^-(f)$. Notice that these definitions depend on the choice of the lift.

Observe that the nonwandering set of f, $\Omega(f)$, is equal to the (not necessarily disjoint) union of $\Omega^+(f)$, $\Omega^-(f)$, and $\pi(\Omega(\widetilde{f}))$.

We will need the following definition from [Fra88a].

Definition 10. Let $f : M \to M$ be a homeomorphism of a surface. A *disk chain* for f is a finite collection of embedded open disks, $U_1, \ldots, U_n \subset M$ satisfying:

1. $f(U_i) \cap U_i = \emptyset$ for all i.
2. For all i, j, either $U_i = U_j$ or $U_i \cap U_j = \emptyset$.
3. For each $i < n$ there exists a positive integer m_i such that $f^{m_i}(U_i) \cap U_{i+1} \neq \emptyset$.

If $U_1 = U_n$ then we say that $U_1, \ldots, U_n$ is a *periodic disk chain*.

Franks proves the following generalization of a theorem of Brouwer (see also [Bro84], [Fat87]).

Theorem 11. [Fra88a] *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism with isolated fixed points. If f has a periodic disk chain, then f has a fixed point of positive index. In particular, if f has a periodic point, then f has a fixed point.*

The following two lemmas are consequences of this theorem.

Lemma 12. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism. If $\Omega(f) \neq \emptyset$ then f has a fixed point. If $\Omega(f)$ consists of more than just fixed points and the fixed points are isolated, then f has a fixed point of positive index.*

Proof. Let $x \in \Omega(f)$. If x is not a fixed point, then there exists an open disk U containing x such that $f(U) \cap U = \emptyset$. Since x is nonwandering there exists $n > 1$ such that $f^n(U) \cap U \neq \emptyset$. Thus $U_1 = U_2 = U$ is a periodic disk chain. By Theorem 11 f has a fixed point, and if the fixed points are isolated, then there is a fixed point of positive index. □

Although not explicitly stated as a result, the following lemma was proved in [Fra88a].

Lemma 13. *Suppose $f : A \to A$ is an orientation preserving homeomorphism of the open annulus that is homotopic to the identity, and let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be a lift of f. If there is a disk $U \subset \widetilde{A}$ that is both positively and negatively returning, then $\widetilde{f}$, and hence f, has a fixed point. If the fixed points are isolated, then there is a fixed point of positive index.*

Proof. Suppose $U \subset \widetilde{A}$ is both a positively and negatively returning disk. So, there exist $n_1, n_2, k_1, k_2 > 0$ such that $\widetilde{f}^{n_1}(U) \cap (U + k_1) \neq \emptyset$ and $\widetilde{f}^{n_2}(U) \cap (U - k_2) \neq \emptyset$. As shown in [Fra88a], $U + k_1$, $U + 2k_1$, $U + 3k_1$, … , $U + k_2k_1$, $U + (k_1 - 1)k_2$, … , $U + 2k_2$, $U + k_2$, U is a periodic disk chain. Thus, by Theorem 11 the conclusions hold. □

We now give our main theorem of this section, a generalization of the Poincaré-Birkhoff-Franks theorem.

Theorem 14. *Suppose $f : A \to A$ is an orientation preserving homeomorphism of the open annulus that is homotopic to the identity, and suppose the nonwandering set of f, $\Omega(f)$, is connected. If there is a lift $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ possessing a positively returning disk and a negatively returning disk both intersecting $\pi^{-1}(\Omega(f))$, then $\widetilde{f}$, and hence f, has a fixed point.*

Proof. Let f and $\widetilde{f}$ be as above. For the sake of contradiction, suppose $\widetilde{f}$ has no fixed point. Since $\widetilde{A}$ is homeomorphic to $\mathbb{R}^2$, Lemma 12 implies that $\Omega(\widetilde{f}) = \emptyset$. By the remark following Definition 9 we know that $\Omega(f) = \Omega^+(f) \cup \Omega^-(f)$. From their definitions it is easy to see that $\Omega^+(f)$ and $\Omega^-(f)$ are open subsets of $\Omega(f)$. Since $\Omega(f)$ is connected it follows that $\Omega^+(f) \cap \Omega^-(f) \neq \emptyset$.

Let $x \in \widetilde{A}$ with $\pi(x) \in \Omega^+(f) \cap \Omega^-(f)$. Then there exists a positively returning disk, U_1 and a negatively returning disk, U_2, both containing x. Let $U \subset U_1 \cap U_2$ be an open disk containing x. Since $\pi(x)$ is nonwandering and since $\Omega(\widetilde{f}) = \emptyset$ the disk U must be either positively or negatively returning. If it is positively returning then U_2 must also be positively returning. Similarly, if U is negatively returning then U_1 must also be negatively returning. Thus either U_1 or U_2 is both positively and negatively returning. By Lemma 13 $\widetilde{f}$ has a fixed point. This is a contradiction. Thus $\widetilde{f}$, and hence f, must have a fixed point. $\square$

It is worth making a few comments about the hypotheses of Theorem 14. First of all, notice that the assumption that $\Omega(f)$ is connected is stronger than we need. If there exist positively and negatively returning disks that intersect the same connected component of $\pi^{-1}(\Omega(f))$ then we could use the same proof to show the existence of a fixed point. Secondly, in the definition of positively and negatively returning disks we assume that $k \neq 0$ for both definitions. One may ask if the existence of returning disks with $k = 0$ could be incorporated in Theorem 14. For instance, if there is a homeomorphism with a positively returning disk and a returning disk with $k = 0$, is there a fixed point? The answer is yes; in fact, there is a fixed point even without the positively returning disk. If there is an open disk U satisfying the definition of the returning disks but with $k = 0$ then $U_1 = U_2 = U$ is a periodic disk chain and thus Theorem 11 guarantees the existence of a fixed point.

Unlike the Poincaré-Birkhoff theorem, our proof can guarantee only one fixed point (not two). Also, unlike in Franks' theorem, this one fixed point may have index zero. We have the following example showing that this may indeed occur. The example is based on one from Carter ([Car82]).

Example 15. Consider the flow on $\widetilde{A}$ shown in Figure 2. Let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be the time-one map of this flow and let $f : A \to A$ be the corresponding map on the open annulus. So defined, f is a bounded homeomorphism with only one fixed point. By Corollary 8 this fixed point must have index zero.

Moreover, the next example illustrates that it is necessary for the positively and negatively returning disks to intersect the lift of the nonwandering set. The positively and negatively returning disks give us reliable twist information only if they have some recurrence.

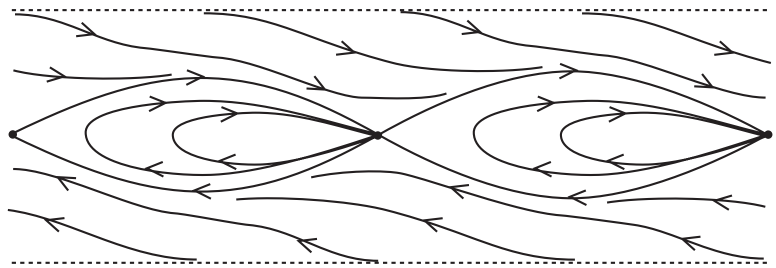

FIGURE 2. The lift of a map with one fixed point of index zero

Example 16. Consider the time-one map, $\widetilde{f} : \widetilde{A} \to \widetilde{A}$, of the flow shown in Figure 3. Let $f : A \to A$ be the corresponding map of the open annulus. The map f is a bounded homeomorphism with a connected nonwandering set. Moreover, $\widetilde{f}$ possesses positively and negatively returning disks. Yet f has no fixed point.

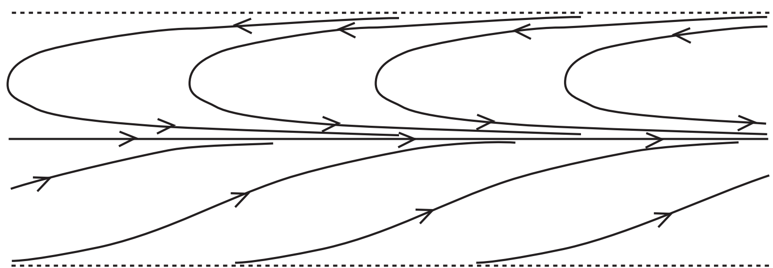

FIGURE 3. A map with positively and negatively returning disks and no fixed points

In Example 16 we see that the fact that a point is in a negatively returning disk does not necessarily imply that the points toward which it tends are in negatively returning disks themselves. However, the converse is true, as the next proposition shows.

Proposition 17. *Let $f : A \to A$ be a homeomorphism of an open or closed annulus and let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be a lift of f. Let $x \in A$. If $\omega(x) \cap \Omega^+(f) \neq \emptyset$ then there is a positively returning disk containing $y \in \pi^{-1}(x)$. If $\omega(x) \cap \Omega^-(f) \neq \emptyset$ then there is a negatively returning disk containing $y \in \pi^{-1}(x)$.*

Proof. Suppose $\omega(x) \cap \Omega^+(f) \neq \emptyset$ and $y \in \pi^{-1}(x)$. Let $z \in \omega(x) \cap \Omega^+(f)$. Then there exists a positively returning disk U such that z is in $\pi(U)$. Also, there exists $n > 0$ such that $f^n(x) \in \pi(U)$. Without loss of generality we may assume that $\widetilde{f}^n(y) \in U$ (if not then translate U by the appropriate integer amount). Since U is a positively returning disk then so is $V = \widetilde{f}^{-n}(U)$. Moreover, V contains y. The case for negatively returning disks is proved similarly. $\square$

In Examples 15 and 16 we see that $\Omega^+(f)$ and $\Omega^-(f)$ are disjoint sets. Example 18 shows that this need not be the case in general. Moreover, we will see that for a point x with $\pi(x) \in \Omega^+(f) \cap \Omega^-(f)$, there may be positively and negatively returning disks containing x that are arbitrarily small.

Example 18. We begin with a rectangle N and create a triple horseshoe by wrapping N around the annulus twice (see Figure 4). Extend f to a homeomorphism on all of A. If desired we may make f bounded. Choose a lift $\widetilde{f}$ as shown in Figure 4.

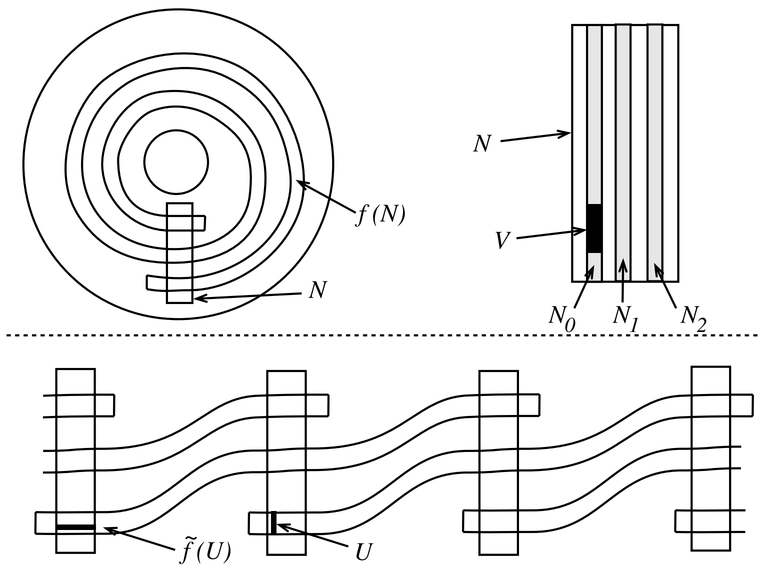

FIGURE 4. A map with $\Omega^+(f) \cap \Omega^-(f) \neq \emptyset$

Inside this triple horseshoe is an invariant set S on which f is conjugate to the full three-shift Σ_3. In particular, let $N_0, N_1, N_2 \subset N$ be the three components of $N \cap f^{-1}(N)$. Then the conjugacy $g : S \to \Sigma_3$ is given by $g(x) = (\dots, a_{-1}, a_0, a_1, \dots)$ where $a_i = j$ if $f^i(x) \in N_j$. Notice that $S \subset \Omega(f)$. Also observe that for points in the lift, a 0 in the itinerary corresponds to movement left and a 2 corresponds to movement right. So, for instance, if $y \in S$ has an itinerary with a finite number of 0s and 1s and $x \in \pi^{-1}(y)$, then $(\widetilde{f}^n(x))_1$ will tend to positive infinity.

Let $y \in S$ be the fixed point with itinerary $(\dots, 0, 0, 0, \dots)$ and let $x \in \pi^{-1}(y)$. We claim that $y \in \Omega^+(f) \cap \Omega^-(f)$ and moreover, every sufficiently small disk containing x is both positively and negatively returning. Let $V \subset N$ be the disk $\mathrm{Int}(N_0 \cap f(N_0))$ and let $U \subset \widetilde{A}$ be the component of $\pi^{-1}(V)$ containing x. Examining the dynamics on $\widetilde{A}$ (see Figure 4) we see that U is negatively returning (with $n = 1$, $k = -1$) and positively returning (with $n = 5$, $k = 1$).

Moreover, we claim that any disk $W \subset U$ containing x is both positively returning and negatively returning. Let $y' = g^{-1}(\dots, a_0, a_1, \dots)$ with $a_i = 0$ for $i < N$ and $i \geq 3N$ and $a_i = 2$ for $N \leq i < 3N$. For N large enough $y', f^{4N-1}(y') \in \pi(W)$. Let $x' \in W \cap \pi^{-1}(y')$. So defined, $\widetilde{f}^{4N-1}(x') \in W + 1$ (according to the itinerary x' moves left $2N - 1$ times and right $2N$ times). Thus, W is positively returning with $n = 4N - 1$ and $k = 1$. It is clear that W is negatively returning with $n = 1$, $k = -1$.

4. Fixed points of bounded homeomorphisms

In this section we investigate fixed points of bounded homeomorphisms of the open annulus. We begin by applying Theorem 14 to this class of homeomorphisms. We then describe the behavior of bounded homeomorphisms possessing one or fewer fixed points.

Theorem 19. *Suppose $f : A \to A$ is a bounded, orientation-preserving homeomorphism of an open annulus that is homotopic to the identity, and suppose $\Omega(f)$ is connected. If there is a lift of f, $\widetilde{f} : \widetilde{A} \to \widetilde{A}$, and points $x, y \in \widetilde{A}$ with $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$ and $\lim_{n\to\infty}(\widetilde{f}^n(y))_1 = \infty$, then $\widetilde{f}$, and hence f, has a fixed point.*

Proof. Suppose $x, y \in \widetilde{A}$ with $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$ and $\lim_{n\to\infty}(\widetilde{f}^n(y))_1 = \infty$. Since f is bounded, $\omega(\pi(y)) \neq \emptyset$. Let $z \in \omega(\pi(y))$, then let $y' \in \pi^{-1}(z)$. If y' is a fixed point then so is z, and we're done. So assume that y' is not fixed. Let $U \subset \widetilde{A}$ be any disk containing y' small enough that $\widetilde{f}(U) \cap U = \emptyset$ and $\pi(U) \subset A$ is a disk. Since z is nonwandering there are infinitely many positive integers n and corresponding integers $k = k(n)$ such that $\widetilde{f}^n(U) \cap (U + k) \neq \emptyset$. Since $z \in \omega(\pi(y))$ and $\lim_{n\to\infty}(\widetilde{f}^n(y))_1 = \infty$, then for n large enough we can guarantee that $k > 0$. Thus, U is a positively returning disk with $U \cap \pi^{-1}(\Omega(f)) \neq \emptyset$. Similarly, since $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$ there is a negatively returning disk intersecting $\pi^{-1}(\Omega(f))$. By Theorem 14 f has a fixed point. □

In [Car82] Carter considers the case where g is a twist homeomorphism of the closed annulus A with at most one fixed point in the interior. She proves that there is an essential simple closed curve C in the interior which intersects its image in at most one point. As we saw in Proposition 7, if g is a bounded homeomorphism of the open annulus, then there are essential simple closed curves which do not intersect their images. Thus it is not clear how one would generalize her theorem for bounded homeomorphisms. We do find that bounded homeomorphisms having having at most one fixed point do have special properties. We present them in Theorem 20. In particular, we see that if f has at most one fixed point then the bad behavior found in Example 18 cannot occur.

We state the following theorem for bounded homeomorphisms of the open or closed annulus. Recall that for the closed annulus every homeomorphism is bounded; thus for the closed annulus, the boundedness hypothesis is redundant.

Theorem 20. *Suppose $f : A \to A$ is an orientation-preserving, bounded homeomorphism of the open or closed annulus that is homotopic to the identity, and suppose f has at most one fixed point. Let $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ be a lift of f. Then, for each $x \in \widetilde{A}$ one of the following is true:*

1. $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = \infty$,
2. $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$, *or*
3. $\lim_{n\to\infty}\widetilde{f}^n(x) = p$ *for some fixed point p of $\widetilde{f}$.*

Moreover, if $\mathrm{Fix}(\widetilde{f}) = \emptyset$ *and* $\Omega(f)$ *is connected, then* $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = \infty$ *for all* $x \in \widetilde{A}$ *or* $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$ *for all* $x \in \widetilde{A}$.

Proof. First, assume that A is the open annulus. Suppose f has at most one fixed point. Let $x \in \widetilde{A}$. Suppose that $\omega(x)$ is not empty and consists of more than a single fixed point. Notice that since the set of fixed points of $\widetilde{f}$ is either empty or discrete (all being integer translates of one another) $\omega(x)$ can't consist of only fixed points. In particular, since $\omega(x) \subset \Omega(\widetilde{f})$, $\Omega(\widetilde{f})$ must consist of more than just fixed points. Lemma 12 states that $\widetilde{f}$ has a fixed point of positive index. But Corollary 8 states that the Lefschetz index of $\mathrm{Fix}(\widetilde{f})$ is zero; this is a contradiction. Thus $\omega(x) = \emptyset$ or $\lim_{n\to\infty} \widetilde{f}^n(x) = p$ for some fixed point p of $\widetilde{f}$.

Now suppose $\omega(x) = \emptyset$. Since f is bounded Proposition 7 states that there is an essential closed annulus $A_0 \subset A$ that is a forward invariant window for f. Let $\widetilde{A}_0 = \pi^{-1}(A_0)$. Notice that for all n sufficiently large $\widetilde{f}^n(x) \in \widetilde{A}_0$. Since we are concerned with the long-term behavior of x, we may assume without loss of generality that $x \in \widetilde{A}_0$. Since $\omega(x) = \emptyset$, for any $M > 0$ there exists $N_M > 0$ such that $|(\widetilde{f}^n(x) - x)_1| > M$ for all $n > N_M$. Thus the orbit of x tends to infinity, negative infinity, or conceivably both. We will show that the last possibility will never occur. Since A_0 is compact there is an $M' > 0$ such that $|(\widetilde{f}(y) - y)_1| < 2M'$ for all $y \in \widetilde{A}_0$. Thus, $(\widetilde{f}^n(x) - x)_1 > M'$ for all $n > N_{M'}$ or $(\widetilde{f}^n(x) - x)_1 < -M'$ for all $n > N_{M'}$. So, it must be the case that $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = \infty$ or $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$.

Lastly, suppose $\mathrm{Fix}(\widetilde{f}) = \emptyset$ and $\Omega(f)$ is connected. From above we see that for $x \in \widetilde{A}$ either $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = \infty$ or $\lim_{n\to\infty}(\widetilde{f}^n(x))_1 = -\infty$. But, by Theorem 19 we know that both cannot occur.

Now, suppose A is the closed annulus. Then let $A' = S^1 \times (-\varepsilon, 1+\varepsilon)$. Extend f to a bounded homeomorphism on A' as follows. If $(x, y) \in S^1 \times (1, 1+\varepsilon)$, then $f(x, y) = f(x, 1) + (0, (y-1)/2)$. Similarly define f on $S^1 \times (-\varepsilon, 0)$. Applying the result for the open annulus we arrive at the desired conclusions. $\square$

5. Periodic orbits and rotation numbers

As indicated in the introduction, bounded homeomorphisms on noncompact spaces behave in many ways like homeomorphisms on compact spaces. In [Fra88b] Franks proves the following result for homeomorphisms of the closed annulus homotopic to the identity: if a point has a given rational rotation number, then there is a periodic point with that same rotation number. The result clearly fails for homeomorphisms of the open annulus. However, it does hold for bounded homeomorphisms.

Below we have a theorem that applies to the open and closed annulus. As mentioned above, the result for the closed annulus was proved by Franks (Corollary 2.5 in [Fra88b]) and Handel [Han]. The proof is modeled on Franks'. However, the results leading up to his proof were different from those presented here (his arguments used the idea of chain recurrence), thus we state both results. In the next two theorems we consider bounded homeomorphisms of the open and closed annulus. Recall that for the closed annulus boundedness is a redundant notion; every homeomorphism of the closed annulus is bounded.

Theorem 21. *Suppose $f : A \to A$ is an orientation-preserving, bounded homeomorphism of the open or closed annulus that is homotopic to the identity. If $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ is a lift of f, and for some $x \in \widetilde{A}$*

$$\liminf \frac{1}{n}(\widetilde{f}^n(x) - x)_1 \leq \frac{p}{q} \leq \limsup \frac{1}{n}(\widetilde{f}^n(x) - x)_1,$$

then f has a periodic point with rotation number p/q.

Proof. Suppose A is the open annulus. Let $x \in \widetilde{A}$ be a point satisfying the hypotheses of the theorem. First, assume that $p = 0$. We will show that $\widetilde{f}$ has a fixed point. For the sake of contradiction, assume that $\widetilde{f}$ has no fixed points. Then by Theorem 20 $\lim(\widetilde{f}^n(y) - y)_1 = \pm\infty$ for all $y \in \widetilde{A}$. Without loss of generality, assume that $\lim(\widetilde{f}^n(x) - x)_1 = \infty$. Since f is bounded $\omega(\pi(x)) \neq \emptyset$; denote this set Λ, and let $\widetilde{\Lambda} = \pi^{-1}(\Lambda)$.

We first show that $\lim(\widetilde{f}^n(y))_1 = \infty$ for every y in $\widetilde{\Lambda}$. Again, Theorem 20 says that $\lim(\widetilde{f}^n(y))_1 = \pm\infty$, so assume for the sake of contradiction that it is $-\infty$ for some y. Then any point y_0 in $\pi^{-1}(\omega(\pi(y)))$ lies in a negatively returning disk (since iterates of $\pi(y)$ return arbitrarily close to $\pi(y_0)$). Therefore y itself lies in a negatively returning disk U, by Proposition 17. Since $y \in \widetilde{\Lambda}$ and $\lim(\widetilde{f}^n(x))_1 = \infty$, U is also positively returning, so by Lemma 13 $\widetilde{f}$ has a fixed point. This contradicts our assumption, so $\lim(\widetilde{f}^n(y))_1 = \infty$ for every y in $\widetilde{\Lambda}$.

We claim that points in $\widetilde{\Lambda}$ may move only a bounded negative distance. That is, there is a $K > 0$ such that $(\widetilde{f}^n(y) - y)_1 > -K$ for all $y \in \widetilde{\Lambda}$ and $n \in \mathbb{Z}^+$. To prove this, define $\widetilde{\Lambda}_{[0,1]}$ to be the set $\{x \in \widetilde{\Lambda} : 0 \leq (x)_1 \leq 1\}$ and $\widetilde{\Lambda}_+$ to be the set $\{x \in \widetilde{\Lambda} : (x)_1 > 0\}$. For each $x \in \widetilde{\Lambda}_{[0,1]}$, there is an integer $n_x > 0$ such that $\widetilde{f}^{n_x}(x) \in \widetilde{\Lambda}_+$. This same n_x works for points in some neighborhood of x, so by compactness there is an $M > 0$ such that for each $x \in \widetilde{\Lambda}_{[0,1]}$, the set $\{x\} \cup \{\widetilde{f}(x)\} \cdots \cup \{\widetilde{f}^M(x)\}$ intersects $\widetilde{\Lambda}_+$. Then the set $\widetilde{\Lambda}_+ \cup \widetilde{f}(\widetilde{\Lambda}_+) \cup \cdots \cup \widetilde{f}^M(\widetilde{\Lambda}_+)$ is forward-invariant. Therefore no point in $\widetilde{\Lambda}_{[0,1]}$ ever moves farther left than $K' = \min\{(x)_1 : x \in \widetilde{\Lambda}_{[0,1]} \cup \widetilde{f}(\widetilde{\Lambda}_{[0,1]}) \cup \cdots \cup \widetilde{f}^M(\widetilde{\Lambda}_{[0,1]})\}$, and we may take $K = -(K' - 1)$.

Next, we claim that there is an $N > 0$ such that $(\widetilde{f}^N(y) - y)_1 > 2$ for all $y \in \widetilde{\Lambda}$. Let

$$U_n = \{\pi(y) \in \Lambda : (\widetilde{f}^n(y) - y)_1 > K + 2\},$$

and note that this implies $(\widetilde{f}^m(y) - y)_1 > 2$ for all $m \geq n$ and all $y \in \pi^{-1}(U_n)$. Notice that U_n is an open subset of Λ. Moreover, since $\lim(\widetilde{f}^n(y) - y)_1 = \infty$ for all $y \in \widetilde{\Lambda}$, $\{U_n\}_{n>0}$ is an open cover of Λ with $U_n \subset U_m$ when $m > n$. Since Λ is compact, there is an $N > 0$ such that $\Lambda = U_N$. This N has the desired property.

Since the orbit of $\pi(x)$ limits upon Λ, for all k sufficiently large $(\widetilde{f}^{N+k}(x) - \widetilde{f}^k(x))_1 > 1$. A telescoping sum shows that

$$(\widetilde{f}^{nN+k_0}(x) - \widetilde{f}^{k_0}(x))_1 > n$$

for some $k_0 > 0$. Thus,

$$\liminf \frac{1}{n}(\widetilde{f}^n(x) - x)_1 > \frac{1}{N},$$

a contradiction. Thus $\widetilde{f}$ has a fixed point.

Now, assume that $p/q \neq 0$. Let $T : \widetilde{A} \to \widetilde{A}$ be the translation $T(x, y) = (x+1, y)$. Let $\widetilde{g} = T^{-p} \circ \widetilde{f}^q$. So defined, $\widetilde{g}$ is a lift of f^q. Moreover, $y \in \widetilde{A}$ is a fixed point of $\widetilde{g}$ iff $\pi(y)$ is a periodic point of f with rotation number p/q. Lastly, observe that

$$\liminf \frac{1}{n}(\widetilde{g}^n(x) - x)_1 \leq 0 \leq \limsup \frac{1}{n}(\widetilde{g}^n(x) - x)_1.$$

Thus, by the argument above $\widetilde{g}$ has a fixed point, and f has a periodic point with rotation number p/q.

Now, suppose A is the closed annulus. As in the proof of Theorem 20 we may extend f to a bounded homeomorphism of the open annulus $A' = S^1 \times (-\varepsilon, 1 + \varepsilon)$ in such a way that no new periodic points are created. Applying the result for the open annulus we find the prescribed periodic point in A. □

Thus, obviously, if a point has a rational rotation number then there is a periodic point with the same rotation number. In fact, we may make the following conclusion. The result for the closed annulus was proved by Franks ([Fra88b]).

Corollary 22. *Suppose $f : A \to A$ is an orientation-preserving, bounded homeomorphism of the open or closed annulus that is homotopic to the identity. If among all the periodic points there are only a finite number of periods, then every point of A has a rotation number.*

References

[AS76] H. Abelson and C. Stanton, *Poincaré's geometric theorem for flows*, J. Differential Geom., **11**(**1**) (1976), 129–131, MR 54 #3765, Zbl 0328.58014.

[Bir13] G. D. Birkhoff, *Proof of Poincaré's geometric theorem*, Trans. Amer. Math. Soc., **14**(**1**) (1913), 14–22, MR 1 500 933.

[Bir25] G. D. Birkhoff, *An extension of Poincaré's last geometric theorem*, Acta Math., **47** (1925), 297–311.

[Bro84] M. Brown, *A new proof of Brouwer's lemma on translation arcs*, Houston J. Math., **10**(**1**) (1984), 35–41, MR 85h:54080, Zbl 0551.57005.

[BN77] M. Brown and W. D. Neumann, *Proof of the Poincaré-Birkhoff fixed point theorem*, Michigan Math. J., **24**(**1**) (1977), 21–31, MR 56 #6646, Zbl 0402.55001.

[Car82] P. H. Carter, *An improvement of the Poincaré-Birkhoff fixed point theorem*, Trans. Amer. Math. Soc., **269**(**1**) (1982), 285–299, MR 84h:54041, Zbl 0507.55002.

[Con78] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, 38, American Mathematical Society, Providence, R.I., 1978, MR 80c:58009, Zbl 0397.34056.

[Fat87] A. Fathi, *An orbit closing proof of Brouwer's lemma on translation arcs*, Enseign. Math. (2), **33**(**3**-**4**) (1987), 315–322, MR 89d:55004, Zbl 0649.54022.

[Fra82] J. Franks, *Homology and dynamical systems*, CBMS Regional Conference Series in Mathematics, 49, American Mathematical Society, Providence, R. I., 1982, MR 84f:58067, Zbl 0497.58018.

[Fra88a] J. Franks, *Generalizations of the Poincaré-Birkhoff theorem*, Ann. of Math. (2), **128**(**1**) (1988), 139–151, MR 89m:54052, Zbl 0676.58037.

[Fra88b] J. Franks, *Recurrence and fixed points of surface homeomorphisms*, Ergodic Theory Dynam. Systems, **8** (Charles Conley Memorial Issue) (1988), 99–107, MR 90d:58124, Zbl 0634.58023.

[Fra88c] J. Franks, *A variation on the Poincaré-Birkhoff theorem*, Hamiltonian dynamical systems (Boulder, CO, 1987), 111–117. Amer. Math. Soc., Providence, RI, 1988, MR 90e:58095, Zbl 0679.58026.

[Gui97] L. Guillou. *A simple proof of P. Carter's theorem*, Proc. Amer. Math. Soc., **125**(**5**) (1997), 1555–1559, MR 97g:54055, Zbl 0863.54033.

[Gün95] B. Günther, *On the homotopic structure of dynamical systems containing global attractors*, Continua (Cincinnati, OH, 1994), Lecture Notes in Pure and Appl. Math., **170**, 227–248. Dekker, New York, 1995, MR 96e:58094, Zbl 0832.54032.

[Han] M. Handel, Untitled preprint from 1989.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza, MR 96c:58055, Zbl 0878.58019.

[Poi12] H. Poincaré, *Sur un théorème de géométrie*, Rend. Circ. Mat. Palermo, **33** (1912), 375–407.

[RW02] David Richeson and Jim Wiseman, *A fixed point theorem for bounded dynamical systems*, Illinois J. Math., **46**(**2**) (2002), 491–495, MR 1 936 931.

[Win88] H. E. Winkelnkemper, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc., **102**(**4**) (1988), 1028–1030, MR 89e:55006, Zbl 0656.54032.

Dickinson College, Carlisle, PA 17013
richesod@dickinson.edu http://www.dickinson.edu/˜richesod

Swarthmore College, Swarthmore, PA 19081
jwisema1@swarthmore.edu http://www.swarthmore.edu/NatSci/jwisema1

This paper is available via http://nyjm.albany.edu:8000/j/2003/9-4.html.