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Dancing paperclips and the geometric influence on magnetization: A surprising result

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An impressive demonstration of magnetism can be accomplished by placing some metal paperclips on a horizontal surface and then applying a vertical magnetic field. As the field is increased, the paperclips suddenly jump up, do a little dance, and then stand at attention. This behavior is related to the more common demonstration of paramagnetic and diamagnetic materials, which consists of hanging a small aluminum (paramagnetic) or glass (diamagnetic) cylinder horizontally in a strong horizontal magnetic field. A paramagnetic cylinder aligns its axis parallel to the applied field while a diamagnetic cylinder aligns its axis perpendicular to the field. This paper investigates these demonstrations by analyzing a magnetic spheroid in a uniform external field. Although this analysis explains the behavior of the paperclips, it predicts that both paramagnetic and diamagnetic cylinders will align themselves parallel to a uniform external field, in contrast to the common demonstration experiment. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

The first time I taught electrodynamics at the advanced undergraduate level, I discovered a surprising demonstration of a magnetic phenomenon that I was eager to share with my students. While fooling around with a pair of Helmholtz coils oriented to provide a uniform vertical magnetic field, I found that metal paperclips that are initially lying down will spontaneously stand on end as the magnetic field is increased. This behavior typically occurs at a field strength of $\sim 10^{-2}$ Tesla (T) and is a dramatic event as the paperclips "scramble" to get onto their "feet." Figure 1 shows a picture after the paperclips are standing at attention. Students are very impressed by this phenomenon.[1] The paperclips repel each other because they are all magnetized in the same direction. This repulsion is easily verified by moving a paperclip around with your hand.

When I first saw this phenomenon, I was so impressed that I immediately showed it to my class. Unfortunately, I had not taken the time to think about what was actually happening and I was unprepared when one of the students asked why the paperclips suddenly stood up. At first I thought it was obvious, but as I stammered for an answer I realized that I didn't really know. By the next class session I was able to give a satisfactory qualitative explanation (which seemed obvious in hindsight), but it wasn't until much later that I also worked out a satisfactory quantitative explanation as well.

It is worth mentioning that this experiment is closely related to a slightly different but relatively standard demonstration experiment designed to show the behavior of paramagnetic and diamagnetic materials.[2] These materials become magnetized in an applied field and the magnetization depends on the applied field. When this relationship is linear, we have $\mathbf{M} = \chi \mathbf{H}$ with $\chi > 0$ for paramagnetic materials and $\chi < 0$ for diamagnetic materials. In a typical experiment a small cylinder of aluminum (paramagnetic) or glass (diamagnetic) is hung horizontally from a thread in a strong horizontal magnetic field. If the material is paramagnetic, the axis of the cylinder aligns parallel to the applied field, and if the material is diamagnetic, the axis of the cylinder aligns perpendicular to the applied field.

Although it is relatively easy to give a qualitative explanation for this phenomenon, a more in depth investigation shows that the situation is much more subtle. One way of explaining this phenomenon is to assume that the cylinders become magnetized along their axes. Then when a paramagnetic cylinder is aligned with the magnetic field, the induced magnetization is such that this cylinder looks like a magnetic dipole that is aligned with the field. Any change in this orientation results in a restoring torque that causes the cylinder to return to its original orientation. A diamagnetic cylinder with its axis aligned in the field looks like a magnetic dipole that is antialigned with the field. This cylinder is in an unstable configuration and any change in orientation will result in a torque that rotates it until its axis is perpendicular to the field.

This qualitative explanation relies on the assumption that the magnetization is along the axis of the cylinder. But when the concept of magnetization is typically introduced in an undergraduate course, it is assumed that the magnetization is parallel (or antiparallel) to the applied field. This assumption means that the cylinders will magnetize along the direction of the applied field regardless of their orientation. If the applied field is uniform, then the magnetization will be uniform and the net magnetic moment of the cylinder will be independent of its orientation. Furthermore, the magnetic moment will be either parallel or antiparallel to the applied field so that there will be no net torque on the cylinder regardless of its orientation. Thus, we immediately see that this demonstration is not as simple as it seems and a correct explanation must include some subtle issues such as a nonuniform applied field, a nonuniform magnetization in the object, or a preference to magnetize along a particular axis of the object.

In this paper I show that a long, ferromagnetic rod has a strong preference to magnetize along its axis. Paramagnetic and diamagnetic rods will also have a directional preference when magnetized but the effect is extremely weak. That is, the direction of magnetization is almost parallel (or antiparallel) to the applied field. Interestingly, while a paramagnetic rod shows a preference to magnetize along its axis, a diamagnetic rod shows a preference to magnetize perpendicular to its axis. This behavior causes both paramagnetic and dia-
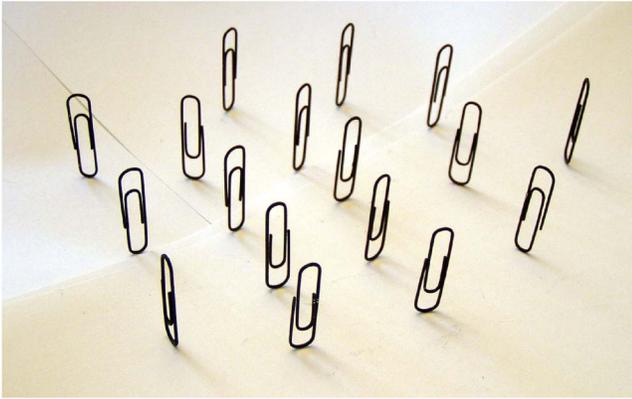
Fig. 1. A picture of standing paperclips (painted black) in the presence of a vertical magnetic field. The paperclips are magnetized and thus repel each other but why do they stand on end?

Fig. 3. Two magnetic dipoles at fixed locations but free to rotate will interact with each other to obtain a preferred (lowest energy) orientation.

magnetic rods to align themselves in a uniform external field, in contrast to what is seen in the standard demonstration. A more careful look at the standard demonstration reveals that the magnetic field is highly nonuniform. Taking this nonuniform field into account allows us to qualitatively understand the behavior of a diamagnetic cylinder.

II. A SIMPLE ANALYSIS

To try to obtain a qualitative understanding as to why an object might prefer to magnetize along a particular axis, let us first examine this problem in a simple manner. We consider a long thin rod in an applied magnetic field and assume that there are microscopic magnets (tiny dipoles) within the material. For a paramagnetic (or ferromagnetic) material these microscopic magnets will try to align in the applied field; the dipoles will antialign in a diamagnetic material. If we consider only the interaction between the dipoles and the applied field (neglecting any dipole-dipole interactions), the material should become magnetized to the same extent regardless of the orientation of the rod. Thus, to understand any preferential alignment we must investigate how the individual dipoles interact with each other. If we consider the rod as one dimensional, then it is easy to see that one configuration is energetically preferred over the other.

Figure 2 shows a schematic of a magnetizable rod with its axis parallel and perpendicular to an applied magnetic field, assuming that there are no dipole-dipole interactions. Because like poles repel and unlike poles attract, the dipole-dipole interactions favor the alignment shown in Fig. 2(a)
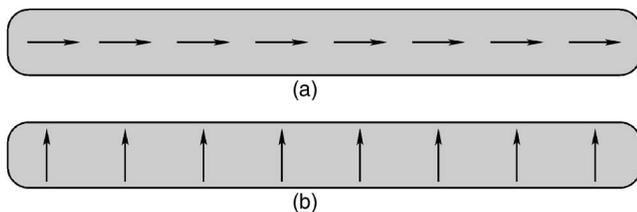

Fig. 2. A magnetizable rod is imagined to consist of a one-dimensional line of dipoles with no dipole-dipole interactions. In (a) the applied field is parallel to the axis of the rod and in (b) the applied field is perpendicular to the axis.
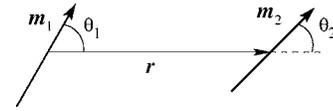
compared to Fig. 2(b). Thus, it should take less energy to orient the dipoles parallel to the rod than it is does to orient them perpendicular to the rod.

This explanation can be made quantitative by considering two dipoles that can rotate freely but are at fixed locations with no external field (see Fig. 3). Beginning with the interaction energy for a magnetic dipole $\mathbf{m}$ in an applied field $\mathbf{B}$, it is straightforward to find the interaction energy between two magnetic dipoles with separation vector $\mathbf{r}$ as

$$U = -\mathbf{m}\cdot\mathbf{B} = \frac{\mu_0}{4\pi r^3}[\mathbf{m}_1\cdot\mathbf{m}_2 - 3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}})], \qquad (1)$$

where μ_0 is the magnetic permeability of free space. Students can be asked to minimize this energy to find the stable configurations of the two interacting dipoles.[3] Not surprisingly, this calculation leads to $\theta_1=\theta_2=0$ (or $\pm\pi$) and $\theta_1=-\theta_2=\pm\pi/2$. The lowest energy state is for $\theta_1=\theta_2=0$. It is not too difficult to imagine that a long line of dipoles in an applied field would then favor the orientation given in Fig. 2(a).

Although this simple model gives a reasonably good picture of why a rod would prefer to magnetize along its axis, it suggests that both paramagnetic and diamagnetic materials will have the same preference. This model neglects how the magnitude of the dipoles depends on the external field. For example, in Fig. 2(a), if the material is paramagnetic, then the applied field points from left to right. When the field of the dipoles is added to the applied field, we see that the magnetic field at the location of each dipole is increased due to the field of its neighbors. This larger field strength will lead to an increase in the magnitude of the dipoles, thereby strengthening their tendency to align. Conversely, if the material is diamagnetic, then the external field points from right to left. In this case, the field of the dipoles leads to a smaller field strength at the location of each dipole. Correspondingly, the magnitude of the dipoles will be decreased, thereby weakening their tendency to align. Therefore, in the case of a diamagnetic rod, the preference for alignment along the axis (due to dipole-dipole interactions) is inhibited by the decrease in the local field. Consequently, it is not obvious—even qualitatively—whether or not a diamagnetic rod will have a preference to magnetize along its axis. Thus, a more complete treatment is necessary to understand how such an object becomes magnetized.

III. A MORE COMPLETE ANALYSIS

A. Magnetized object with no external field

Before beginning a more complete calculation of a magnetizable object in an applied field, it is useful to first consider a uniformly magnetized object with no externally applied field. A standard undergraduate problem is to determine the magnetic field of a uniformly magnetized sphere (or the

equivalent problem of the electric field of a uniformly polarized sphere). For a magnetization $\mathbf{M}$, the fields inside the sphere are (surprisingly) uniform:[4]

$$\mathbf{B} = \tfrac{2}{3}\mu_0\mathbf{M} \quad \text{and} \quad \mathbf{H} = -\tfrac{1}{3}\mathbf{M}. \qquad (2)$$

Note that $\mathbf{H}$ points in the opposite direction of the magnetization $\mathbf{M}$. For this reason, it is often referred to as a *demagnetizing* field. This demagnetization effect is usually described by the demagnetizing factors n_x, n_y, n_z defined by $H_x = -n_x M_x$, $H_y = -n_y M_y$, and $H_z = -n_z M_z$.[5] The demagnetizing factors are dimensionless geometry-dependent quantities between 0 and 1 such that $n_x + n_y + n_z = 1$. For a sphere, Eq. (2) (and symmetry) dictates that $n_x = n_y = n_z = 1/3$. An infinitely long cylinder with its axis along the x direction has $n_x = 0$ and $n_y = n_z = 1/2$, and an infinite slab of finite thickness in the x direction has $n_x = 1$ and $n_y = n_z = 0$. These values can be verified by using the magnetic boundary conditions and symmetry to find the fields of a uniformly magnetized cylinder or slab. (The demagnetizing factors for ellipsoids are given in the Appendix.)

The demagnetizing factors tell us how large a demagnetizing field we can expect to find inside the object. This demagnetizing field is related to how much energy is required to magnetize the object. The energy required to assemble a magnetized domain assuming there is no externally applied magnetic field is given by[6]

$$E_m = -\frac{1}{2}\mu_0 \int \mathbf{M} \cdot \mathbf{H}\, dV, \qquad (3)$$

where the integration is over the volume of the object V, and $\mathbf{H}$ is the field arising from the magnetization $\mathbf{M}$. If the magnetization is directed along one of the principal axes of the object, then $\mathbf{H} = -n\mathbf{M}$, where n is the demagnetizing factor associated with the particular axis. If we substitute this relation into Eq. (3), we find

$$E_m = \tfrac{1}{2}\mu_0 n M^2, \qquad (4)$$

which implies that for a given magnetization, the lowest energy state corresponds to magnetizing the object along the axis with the smallest demagnetizing factor (or the smallest demagnetizing field). For an infinite cylinder, this axis is along the axis of the cylinder ($n_x = 0$ compared to $n_y = n_z = 1/2$), and for an infinite slab of finite thickness this axis will be parallel to the slab ($n_y = n_z = 0$ compared to $n_x = 1$).

It is not trivial to determine the demagnetizing factors except for a few symmetric objects. However, there is a relatively easy way of determining qualitatively whether the demagnetizing factor is large or small. For uniformly magnetized objects, we can calculate $\mathbf{H}$ from a fictitious magnetic surface charge density given by $\sigma_m = \mathbf{M} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit normal to the surface of the object.[7] Thus, if we look at the object in terms of its surface charges, it is easy to see when there will be a large internal field rather than a small internal field. For example, a cylinder with a radius large compared to its length looks like a capacitor with closely spaced plates. This geometry will give rise to a relatively large internal field and therefore a demagnetizing factor close to unity.[8] On the other hand, a cylinder that is long compared to its radius will be a capacitor with plates that are very far apart. This geometry will give rise to a much smaller internal field and therefore a demagnetizing factor close to zero. Thus, we can see that it will take less energy to magnetize a long cylinder along its axis than to give it an equiva-

lent magnetization perpendicular to its axis. On the contrary, a disk will be more difficult to magnetize along its axis than to magnetize it perpendicular to its axis.

B. Magnetizable object in an external field

Let us now consider a magnetizable object in an external field. We will consider an ellipsoid composed of a homogeneous linear material in a uniform external magnetic field. The restriction to linear materials is made for simplicity, but even ferromagnetic materials are typically linear (with initial relative permeabilities ranging from 10–10^4) as long as the applied field is very small.[9] As for a sphere, a magnetizable ellipsoid in an applied field can be solved exactly and results in uniform fields (and magnetization) inside the object. A long thin rod and a flat plate are two limiting cases that are easily approximated by choosing appropriate parameters for the ellipsoid. These geometries are also easily solved for isotropic and homogeneous ferromagnetic materials.[10]

We first note that another standard problem in an intermediate level course in electromagnetism is a magnetic sphere in a uniform magnetic field (or the equivalent problem of a dielectric sphere in a uniform electric field). When written in terms of demagnetizing factors, the solution for an ellipsoid looks exactly like the solution for a sphere. For an applied field given by $\mathbf{B}_0$ along one of the principal axes, the ellipsoid obtains internal fields of

$$\mathbf{B} = \mathbf{B}_0 + \mu_0(1 - n)\mathbf{M} \qquad (5)$$

and

$$\mathbf{H} = \mathbf{H}_0 - n\mathbf{M}, \qquad (6)$$

where n is the demagnetizing factor for the object along the particular axis. Note that Eqs. (5) and (6) are valid for any relation between $\mathbf{B}$ and $\mathbf{H}$; if they are linearly related by $\mathbf{B} = \mu\mathbf{H}$ where μ is the magnetic permeability of the material, then the magnetization is given by

$$\mu_0\mathbf{M} = \left(\frac{\chi}{1 + n\chi}\right)\mathbf{B}_0, \qquad (7)$$

where χ is the magnetic susceptibility.[11]

Note that a change in the demagnetizing factor n has a qualitatively different effect on the magnetization $\mathbf{M}$ for paramagnetic and diamagnetic materials. Equation (7) shows that increasing n for a paramagnetic material leads to a decrease in the magnitude of the magnetization M. Conversely, increasing n for a diamagnetic material leads to an increase in M. As we shall see, this behavior affects the direction of the net magnetization and the resulting torque on the object.

We next consider a spheroid (see Appendix) with its axis of symmetry along the x axis at an angle θ with respect to the applied field (see Fig. 4). Prolate spheroids are cigar shaped and have $n < 1/3$ along the symmetry axis and oblate spheroids are disk shaped and have $n > 1/3$ along the symmetry axis. When the external field is not applied along one of the principal axes, the magnetization will still be uniform but it will no longer be in the same direction as the applied field. In this case, the components of magnetization are given by

$$\mu_0 M_i = \left(\frac{\chi}{1 + n_i\chi}\right)B_{0i}, \qquad (8)$$

where $i = 1, 2, 3$ represents the x, y, z components, respectively. The different values of the demagnetization factors
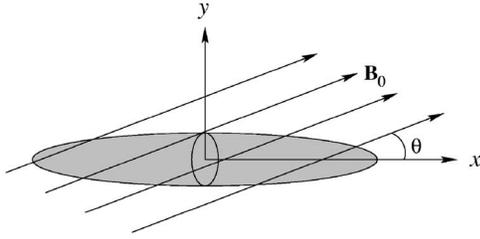
Fig. 4. A uniform magnetic field is applied at an angle θ with respect to the object's axis of symmetry. Because of the different demagnetizing factors in the x and y directions, the magnetization will not (in general) be in the same direction as the applied field.

lead to a difference between the direction of the applied field and the direction of the object's magnetization. The result is a torque acting on the object.

Before continuing, let us briefly consider the direction of the magnetization. For paramagnetic and diamagnetic materials, the susceptibility is typically on the order of 10^{-5}. Thus, for any value of n_i, Eq. (8) gives $\mu_0 M_i \approx \chi B_{0i}$. If we let α be the angle between the object's magnetization and the x axis, we have

$$\alpha = \arctan\left(\frac{M_y}{M_x}\right) \approx \arctan\left(\frac{B_{0y}}{B_{0x}}\right). \tag{9}$$

Thus we see that the direction of magnetization for a paramagnetic object is essentially aligned with the applied field $\alpha \approx \theta$. For a diamagnetic object $\chi < 0$, so M_x and M_y have signs opposite to B_x and B_y, respectively. Therefore, $\alpha \approx \theta + \pi$ and the magnetization is essentially antialigned with the field. By using Eq. (8), the angle between the applied field and magnetization is found to be less than 0.0001° for a typical paramagnetic or diamagnetic object with $n_x \approx 0$ and $n_y \approx 1/2$ (very prolate). This small angular difference will lead to a very small torque even for quite strong magnetic fields.

A ferromagnetic material has $\chi \gg 1$ so that Eq. (8) gives $\mu_0 M_i \approx B_{0i}/n_i$ (as long as $n_i \neq 0$). Thus,

$$\alpha = \arctan\left(\frac{M_y}{M_x}\right) \approx \arctan\left(\frac{n_x B_{0y}}{n_y B_{0x}}\right), \tag{10}$$

and as long as we are dealing with a very prolate object, $\alpha \ll 1$ unless $B_{0y} \gg B_{0x}$. For the extreme case of an infinite cylinder ($n_x = 0$ and $n_y = 1/2$), we have $M_y/M_x \approx 2B_y/\chi B_x$, so we obtain the same behavior that $\alpha \ll 1$ unless $B_{0y} \gg B_{0x}$. The fact that $\alpha \ll 1$ means that a very prolate ferromagnetic object will be magnetized almost completely along its axis of symmetry unless the field is applied nearly perpendicular to this axis. For example, using Eq. (8) with $\chi = 200$ gives a magnetization angle of only $\alpha \approx 7°$ for an applied field with $\theta \approx 81°$. This strong preference to magnetize along the axis will lead to a very large torque that tends to align the object with the direction of the field.[12]

We now turn our attention to the energy of a magnetizable object in an applied field. Similar to Eq. (3), this energy can be written as[13]

$$E_m = -\frac{1}{2}\mu_0 \int \mathbf{M} \cdot \mathbf{H} \, dV, \tag{11}$$

but now $\mathbf{M}$ is the induced magnetization and $\mathbf{H}$ is the field that would have been present in the absence of the object

(the applied field).[14] Because the applied field and magnetization are both constant in our problem, the integration in Eq. (11) is trivial. With $\mathbf{B}_0 = B_0 \cos\theta\, \hat{\mathbf{e}}_x + B_0 \sin\theta\, \hat{\mathbf{e}}_y$ and the components of magnetization given in Eq. (8), we obtain

$$E_m = -\frac{B_0^2}{2\mu_0} V\chi\left(\frac{\cos^2\theta}{1 + n_x\chi} + \frac{\sin^2\theta}{1 + n_y\chi}\right). \tag{12}$$

The torque on the object can be calculated by taking the cross product of the magnetic moment with the applied field or by differentiating Eq. (12) with respect to θ.[15] The result is

$$\boldsymbol{\tau}_m = (\mathbf{M}V) \times \mathbf{B}_0 = \frac{dE_m}{d\theta}\hat{\mathbf{e}}_z \tag{13a}$$

$$= \frac{B_0^2 V\chi^2(1 - 3n)\sin\theta\cos\theta}{\mu_0[1 + n\chi][2 + (1 - n)\chi]}\hat{\mathbf{e}}_z, \tag{13b}$$

where $n = n_x$ is the demagnetizing factor along the axis of symmetry and we have used the fact that $n_y = n_z = (1 - n)/2$.[16] Equation (13b) is relatively easy to understand qualitatively once we realize that the denominator is always positive if $\chi > -1$ (which holds for all known materials). The sign of the torque is then governed by the factor $(1 - 3n)\sin\theta\cos\theta$. Prolate objects ($n < 1/3$) experience a torque causing their axis of symmetry to align with the external field and oblate objects ($n > 1/3$) experience a torque causing their axis of symmetry to align perpendicular to the external field. It is easy to verify that $\theta = 0$ is an energy minimum for prolate objects and $\theta = \pi/2$ is an energy minimum for oblate objects. In addition, a sphere ($n = 1/3$) will experience no torque in an applied field as expected.

If you have not already realized, the most surprising aspect of this result is that it applies to both diamagnetic materials and paramagnetic materials. Consequently, prolate spheroids made of a paramagnetic or diamagnetic material will align their symmetry axis *parallel* to an external (uniform) magnetic field. I will now show how this result makes good physical sense.

First, consider a paramagnetic sphere in which the demagnetizing field is antiparallel to the applied field. (For a diamagnetic sphere the demagnetizing field is parallel to the applied field.) Recall that the demagnetizing factor n is a measure of the strength of the demagnetizing field. Thus, for a paramagnetic object a larger value of n means a larger demagnetizing field and therefore a smaller total field and a smaller magnetization. On the contrary, for a diamagnetic object a larger value of n means a larger demagnetizing field and therefore a larger total field and a larger magnetization. This result can be understood mathematically from Eq. (7). For a given applied field $\mathbf{B}_0$, increasing the value of n leads to a smaller $\mathbf{M}$ for a positive χ and a larger $\mathbf{M}$ for a negative χ.

Next, consider a prolate object as shown in Fig. 4 and assume for simplicity that $\theta = \pi/4$ so that the x and y components of the applied field are equal. If the demagnetizing factors are the same in the x and y directions, the magnetization would be parallel (paramagnetic) or antiparallel (diamagnetic) to the applied field and there would be no torque on the object. But for a prolate object $n_x < n_y$. Thus, a paramagnetic material will have $M_x > M_y$, resulting in a magnetic moment that points between $\theta = 0$ and $\theta = \pi/4$. The direction of this magnetic moment results in a torque that tends to align the symmetry axis with the applied field. However, a

diamagnetic material will have $|M_x| < |M_y|$ (both M_x and M_y will be negative) resulting in a magnetic moment that points between $\theta = 5\pi/4$ and $\theta = 3\pi/2$. Again, the direction of the magnetic moment results in a torque that tends to align the symmetry axis with the applied field.

This analysis shows that—contrary to the standard demonstration—a long, thin diamagnetic rod should align itself in a uniform magnetic field. Demonstrating this phenomenon should be simple in principle; a strong uniform magnetic field is all that is needed. Unfortunately, I am unaware of any experimental observation of this phenomenon.

If prolate diamagnetic objects tend to align their symmetry axes in an applied field, why does the standard demonstration for paramagnetic and diamagnetic objects show diamagnetic cylinders aligning perpendicular to the field? My analysis above is based on spheroids in uniform applied fields. The standard demonstration uses small cylinders between two neodymium magnets. In our experiment,[2] the cylinders are approximate 32 mm long and 6 mm in diameter. To accommodate these cylinders, the magnets need to be separated by about 5 cm. These cylindrical magnets have a radius of less than one centimeter. Thus, as the gap increases beyond a centimeter, the field will become increasingly weak at the midpoint between the magnets. As an example, the Pasco instruction manual states that a gap spacing of 8.9 cm gives a field strength at the midpoint that is over 50 times weaker than immediately in front of the magnets—a very nonuniform field.

To consider the effects of a nonuniform field, recall that a magnetic dipole in an external field will experience a force given by $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$. We have already determined that the direction of magnetization is essentially parallel (antiparallel) to the field direction for paramagnetic (diamagnetic) materials. Thus, the magnetic force on such a dipole will be $\mathbf{F} = \pm m \nabla B$, where the positive (negative) sign is for dipoles parallel (antiparallel) to the field. Therefore, a dipole that is parallel to the field will be drawn into regions of high field strength. Conversely, a dipole that is antiparallel to the field will be expelled from regions of high field strength. Therefore, the ends of a paramagnetic cylinder will be attracted to the poles of the magnets while the ends of a diamagnetic cylinder will be repelled from the poles of the magnets. Thus, it is relatively easy to understand the behavior of the standard demonstration as long as the effects of the nonuniform field are included.

So far we have demonstrated that there are two possible effects that result in torques acting on a rod in a magnetic field—one due to a uniform applied field and one due to a nonuniform applied field. Which of these effects is larger? Equation (13b) gives the torque on an object in a uniform external field. For a very prolate object ($n \approx 0$) and a typical paramagnetic or diamagnetic material, the magnitude of the torque due to a uniform field is approximately given by

$$\tau_u \approx B_0^2 V \chi^2 \sin\theta \cos\theta / 2\mu_0. \tag{14}$$

To estimate the torque due to a nonuniform field, we assume that the rod consists of two magnetic dipoles located on the ends of the rod. These dipoles are assumed to be parallel (or antiparallel) to the field with magnitudes $m \approx MV/2$, where M is the magnitude of the magnetization given in Eq. (7). The torque on the rod will then be given by $\tau = 2(L/2)F_{\text{mag}} \sin\theta$, where the magnetic force is approximately given by $F_{\text{mag}} = m\nabla B$. If we estimate the gradient

of the field as $\nabla B \approx B_0/(L/2)$, then the magnitude of the torque due to a nonuniform field will be approximately given by

$$\tau_{n\text{-}u} \approx B_0^2 V \chi \sin\theta / \mu_0. \tag{15}$$

Thus, we see that the torque from the nonuniform field is greater than the torque from the uniform field by a factor of $\approx 2/\chi \cos\theta$. For typical paramagnetic or diamagnetic materials, this factor is approximately five orders of magnitude. Therefore, unless the field is very uniform, $\tau_{n\text{-}u}$ will dominate τ_u showing that the behavior in the standard demonstration is almost certainly due to the nonuniform field of the magnets. This estimate also suggests that to set up an experiment to observe the interesting fact that a diamagnetic cylinder should align itself parallel to a uniform field would require a very strong magnetic field. To obtain a torque that is similar in strength to the standard demonstration (which is quite weak) would require a magnetic field about 100 times stronger than the field near the pole of the magnet (about 0.4 T). Thus, a field of approximately 40 T would be required. Producing such a strong uniform field over a relatively large region of space is nontrivial and probably explains why this phenomenon has not been observed.[17]

C. Magnetizable object in an external field with gravity: The dancing paperclips

It is now relatively straightforward to extend our results to explain the behavior of the "dancing" paperclips described in Sec. I; the only thing missing is gravity. We assume the external field is applied vertically and the object's axis of symmetry defines the x axis at an angle θ with respect to the applied field (see Fig. 4).[18] The geometry of the spheroid makes the calculation of the gravitational potential energy cumbersome. For simplicity we will consider a very prolate object; the object is long and thin such that its aspect ratio $m = a/b \gg 1$. This geometry allows us to approximate the demagnetizing factors as $n_x \approx 0$ and $n_y \approx 1/2$. We write the mass density as ρ and the length of the rod as $L = 2a$. The gravitational potential energy will then be given by $E_g \approx \rho V g a \cos\theta$. If we add this energy to the magnetic energy given in Eq. (12) and divide by $\rho V g a$, we obtain a total (dimensionless) energy for a prolate object of

$$\widetilde{E}_p = \cos\theta - C\chi\left(\frac{2 + \chi \cos^2\theta}{2 + \chi}\right), \tag{16}$$

where $C = B_0^2/(\mu_0 \rho g L)$ is a dimensionless number that gives the relative strengths of the magnetic and gravitational forces.[19] Because μ_0, ρ, g, and L are assumed constant for a given experiment, the number C is essentially a measure of magnetic field strength. We will therefore refer to C as the *magnetic* number.

For a given χ the general behavior of the total energy given by Eq. (16) is simple and is summarized in Fig. 5. When there is no magnetic field ($C = 0$), the energy is just given by a cosine curve. This function has a maximum at $\theta = 0$ and minima at $\theta = \pm \pi/2$.[20] As the magnetic field is increased ($C > 0$), the point $\theta = 0$ undergoes a pitchfork bifurcation from a maximum to a (local) minimum while the points at $\theta = \pm \pi/2$ remain minima. Thus there are now local maxima that act as energy barriers between the minimum energy states. At this point, a prolate object will be stable if
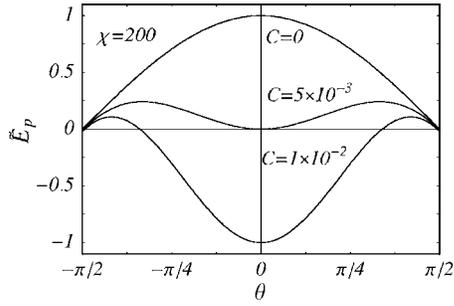
Fig. 5. The (dimensionless) total energy of a very prolate ferromagnetic object with $\chi = 200$ for different magnetic field strengths as given by Eq. (16). The maximum at $\theta = 0$ undergoes a pitchfork bifurcation and becomes a (local) minimum as the magnetic number C is increased. The energy barrier between the minima at $\theta = 0$ and $\theta = \pm \pi/2$ becomes smaller as C is increased further.

lying on the table or if stood on end. This behavior is easily verified experimentally. As the magnetic field is increased further, the minimum at $\theta = 0$ becomes deeper and the barrier between this minimum and the minima at $\theta = \pm \pi/2$ becomes smaller as the local maxima move closer to $\pm \pi/2$. Of course, the paperclips are not sitting exactly at $\pi/2$, only close to it. Thus, at some point the paperclips find themselves on the other side of the barrier allowing them to spontaneously stand on end.

It is interesting to determine the critical value of the field at which the bifurcation occurs ($\theta = 0$ changes from being a minimum to a maximum). Differentiating Eq. (16) twice and setting $\theta = 0$ will tell us whether $\theta = 0$ is a maximum or minimum. The zero of the first derivative gives the critical point for a prolate object as

$$C_p^* = \frac{2 + \chi}{2\chi^2}. \tag{17}$$

From the definition of C, Eq. (17) tells us that the critical magnetic field grows as the square root of the object's length $B_0^* \propto \sqrt{L}$. Although this dependence seems easy to check experimentally, it is not simple for several reasons. First, unless you have a very strong magnetic field constant over a large region of space, you must use ferromagnetic materials which are subject to hysteresis effects. Second, these results are strictly valid only for very prolate spheroids or infinitely long cylinders which are not easy to obtain (this approximation simplified the analysis considerably). Nevertheless, to make some connection with the real world, we made some very crude measurements using small (ACCO #1 trombone style) paperclips. Beginning with a relatively strong magnetic field so a paperclip will be stable when standing upright, we slowly decrease the field strength and note when the paperclip falls over. A series of 10 trials using a fresh paperclip each time gave a fairly consistent critical field value of $B_0^* \sim 2.7 \times 10^{-3}$ T. For a paperclip of length $L \sim 3$ cm and density $\rho \sim 8 \times 10^3$ kg/m^3, the magnetic number $C \sim 2.5 \times 10^{-3}$, giving a magnetic susceptibility of $\chi \approx 200$. Most ferromagnetic materials behave linearly for small applied fields with typical initial susceptibilities ranging from 10 to 10^4 so this result is reasonable.[9] An extensive search[21] revealed that paperclips are typically made from low carbon (mild) steel which has an initial susceptibility of around 150–250.

It is easy to show that the energy barrier in Fig. 5 never completely disappears as the magnetic field is increased. By differentiating Eq. (16), we find that the local maxima occur when

$$\cos \theta = \frac{2 + \chi}{2C\chi^2}. \tag{18}$$

Thus, we see that as $C \to \infty$ the local maxima approach $\theta \to \pm \pi/2$. Therefore, the angles $\theta = \pm \pi/2$ will remain local minima for all finite applied fields. Having said that, the energy barriers will become arbitrarily small for large applied fields. For example, if $\chi \gg 1$, we see from Eq. (18) that $\cos \theta \approx 1/2C\chi$ so that

$$\theta \approx \pm \pi/2 \quad \text{when } C \gg 1/2\chi. \tag{19}$$

For $\chi = 200$ Eq. (19) suggests that we should see spontaneous standing when $C \gg 2.5 \times 10^{-3}$. We did a series of experiments with paperclips and observed that spontaneous standing typically occurs for a field strength of $B_0 \sim 10^{-2}$ T.[22] This field corresponds to a magnetic number of $C \sim 2.8 \times 10^{-2}$ which is consistent with Eq. (19).

In an analogous manner to the calculations for a very prolate object, we can also determine the behavior of a very oblate (disklike) object ($m = a/b \ll 1$) with $n_x \approx 1$ and $n_y \approx 0$. In this case the length of the object is the diameter of the disk $L = 2b$, and because the axis of symmetry lies along the x axis, the gravitational energy is given by $E_g \approx \rho V g b |\sin \theta|$. The results are similar to those for a prolate object except that $\theta = 0$ when the disk is lying down and $\theta = \pi/2$ is the angle that switches from being an energy maximum to an energy minimum causing the disk to stand on end.[23] For completeness, the expressions equivalent to Eqs. (16) and (17) are

$$\tilde{E}_o = |\sin \theta| - C\chi \left(\frac{1 + \chi \sin^2 \theta}{1 + \chi} \right) \tag{20}$$

and

$$C_o^* = \frac{1 + \chi}{2\chi^2}. \tag{21}$$

It might seem surprising that the critical field for an oblate object is slightly smaller than for a prolate object. This result can be traced to the larger difference between n_x and n_y for an oblate object compared to a prolate object.[24] However, for the large susceptibilities typical of ferromagnetic materials, it would be very difficult to experimentally determine any difference between Eqs. (17) and (21).

IV. CONCLUSION

When a vertical magnetic field is applied to paperclips that are horizontal, there is a field for which the paperclips spontaneously stand on end. This behavior makes a wonderful demonstration for an undergraduate course in electrodynamics. The main goal of this paper was to investigate this phenomenon qualitatively and quantitatively by modeling the paperclips as prolate spheroids in a uniform magnetic field. By calculating the energy of this geometry, it was shown that a rod standing on end becomes a stable configuration at a critical value of the magnetic field. A simple experiment was performed that gave reasonable agreement with the theory.

Surprisingly, this analysis shows that both paramagnetic and diamagnetic objects behave similarly and in the absence of gravity, these cylinders will align parallel to a uniform magnetic field. This behavior contradicts the standard demonstration that shows diamagnetic cylinders align perpendicular to an applied field. This observation can be explained by noting that the applied field is highly nonuniform in the standard demonstration. Thus, the rotation of the cylinders is a direct result of the forces due to the nonuniform field.

APPENDIX: DEMAGNETIZING FACTORS FOR ELLIPSOIDS AND SPHEROIDS

Consider an ellipsoid whose axial directions coincide with a Cartesian coordinate system defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{A1}$$

The semiaxes a, b, and c give the radius of the ellipsoid along the x, y, and z axes, respectively. The demagnetizing factor n_x for an ellipsoid is given by[25]

$$n_x = \frac{1}{2}abc \int_0^\infty \frac{ds}{(s+a^2)R_s}, \tag{A2}$$

where $R_s = \sqrt{(s+a^2)(s+b^2)(s+c^2)}$; the expressions for n_y and n_z are obtained by replacing the denominator in Eq. (A2) by $(s+b^2)R_s$ and $(s+c^2)R_s$, respectively. We are interested in prolate (cigar shaped) or oblate (disk shaped) spheroids which have $a>b=c$ or $a<b=c$, respectively. The demagnetizing factors for these objects are as follows.[26]

Prolate spheroid ($m=a/b>1$),

$$n_x = \frac{1}{(m^2-1)}\left[\frac{m}{2\sqrt{m^2-1}} \ln\left(\frac{m+\sqrt{m^2-1}}{m-\sqrt{m^2-1}} \right) - 1 \right], \tag{A3}$$

and $n_y=n_z=(1-n_x)/2$.

Oblate spheroid ($m=a/b<1$),

$$n_x = \frac{m^2}{2(1-m^2)}\left[\frac{1}{m\sqrt{1-m^2}} \arcsin\sqrt{1-m^2} - 1 \right], \tag{A4}$$

and $n_y=n_z=(1-n_x)/2$.

Figure 6 shows the demagnetizing factors for prolate and oblate spheroids as given by Eqs. (A3) and (A4). As is clear in Fig. 6, the limiting cases of a flat plate ($m\to 0$) and a long thin rod ($m\to\infty$) give the expected demagnetizing factors of $n_x=1$, $n_y=n_z=0$ and $n_x=0$, $n_y=n_z=1/2$, respectively.
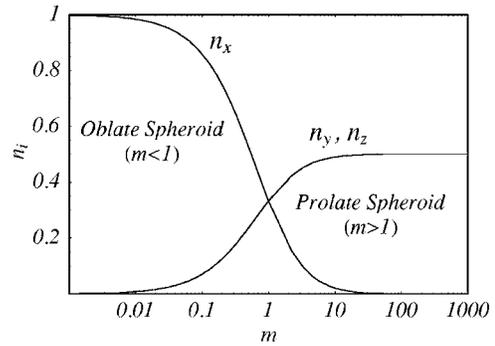
Fig. 6. Demagnetizing factors for oblate ($m=a/b<1$) and prolate ($m=a/b>1$) spheroids as calculated from Eqs. (A3) and (A4).

tinctly remember that this tendency to align in the field is so strong (for ferromagnetic objects) that it was virtually impossible to hold a long (4–5 in.) iron bolt perpendicular to the field.

[2] We use a magnetic demonstration system available from Pasco Scientific, item EM-8644A. This demonstration consists of a variable gap magnet and various accessories for demonstrating magnetic forces and eddy currents.

[3] Although no connection is made to the experiments discussed here, this problem is discussed by D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, Upper Saddle River, NJ, 1999). Problem 6.21, pp. 281–282.

[4] See, for example, Ref. 3, pp. 264–265.

[5] We are assuming that the coordinate axes are aligned with the principal axes of the object. In general the demagnetizing effect is described by a demagnetizing factor tensor and n_x, n_y, and n_z are the principal values of this tensor. See L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon, New York, 1984).

[6] C. Kittel, "Theory of the structure of ferromagnetic domains in films and small particles," Phys. Rev. **70**, 965–971 (1946).

[7] See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), pp. 193–194.

[8] Technically speaking, demagnetizing factors are only defined for objects that have uniform internal fields.

[9] See Ref. 7, p. 190.

[10] The magnetization can still be expressed as $M=\chi'H$ where χ' is no longer constant but a field-dependent proportionality factor. Therefore, the results for a ferromagnetic sphere can be obtained from the results for a non-ferromagnetic sphere by replacing χ by χ'. See E. M. Pugh and E. W. Pugh, *Principles of Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1960), pp. 289–292.

[11] Recall that the magnetic susceptibility is defined by a linear relation between the magnetization and the applied field $\mathbf{M}=\chi\mathbf{H}$; $\chi>0$ for paramagnetism and $\chi<0$ for diamagnetism. The magnetic permeability μ of an object is related to its susceptibility by $1+\chi=\mu/\mu_0$. The quantity $\mu_r=\mu/\mu_0$ is the relative permeability.

[12] Note that this strong preference to magnetize along the axis of a long ferromagnetic rod will make it difficult to hold such an object perpendicular to a strong external field. See comment in Ref. 1.

[13] R. E. Rosensweig, *Ferrohydrodynamics* (Cambridge U.P., New York, 1985), p. 98.

[14] Note that this expression for the energy does not include the work done by the sources against the induced electromotive forces. For a brief discussion of this point, see Ref. 7, p. 216.

[15] Our definition of θ is such that when θ is increasing, the object is rotating in the negative $\hat{\mathbf{e}}_z$ direction.

[16] The corresponding result for a dielectric ellipsoid in an applied electric field is similar. See V. V. Batygin and I. N. Toptygin, *Problems in Electrodynamics* (Academic, London, 1964), pp. 44 and 229.

[17] The estimates in this section are only order-of-magnitude estimates and should not be taken too seriously. Nevertheless, these estimates indicate that a fairly strong uniform field would be necessary to observe the alignment of a typical diamagnetic cylinder.

[18] Due to the presence of the horizontal surface on which the paperclips lie, the angle θ is confined to lie between $-\pi/2$ and $\pi/2$.

[19] This dimensionless number is proportional to the Cowling number

[1] In hindsight, I can remember a related phenomenon that I experienced as a graduate student. We used to bring in nonscience students to briefly encounter a very strong magnetic field from an old cyclotron. I can dis-

(which is related to the Alfvén or Kármán numbers) that arises when studying magnetohydrodynamics. See for example, K. Ražnjević, *Physical Quantities and the Units of the International System (SI)* (Begell House, New York, 1995), p. 185.

[20] The minima at $\pm\pi/2$ are not "functional" minima in the sense that Eq. (16) has minima at these point. Instead, these minima occur because the angle θ is restricted to lie between $-\pi/2$ and $\pi/2$. That is, we are dealing with "boundary value" minima.

[21] The manufacturer responded to our request for information on the composition of their paperclips by saying it was proprietary information.

[22] The field strength at which spontaneous standing takes place depends on whether the paperclips have been used and whether they are sitting on a flat smooth surface. Old paperclips on a rough surface will spontaneous stand at a lower field strength than new paperclips on a perfectly smooth surface.

[23] Spontaneous standing for disks is more difficult to achieve than for paperclips. Presumably, this difficulty is due to the fact that the disks lie much more flatly against the table than do the paperclips.

[24] This larger difference between n_x and n_y can be traced to the effective dimension of the object. A long thin rod is essentially one dimensional while a flat plate is two dimensional.

[25] See Ref. 5, p. 24.

[26] L. Sun, Y. Hao, C.-L. Chien, and P. C. Searson, "Tuning the properties of magnetic nanowires," IBM J. Res. Dev. **49**, 79–102 (2005).
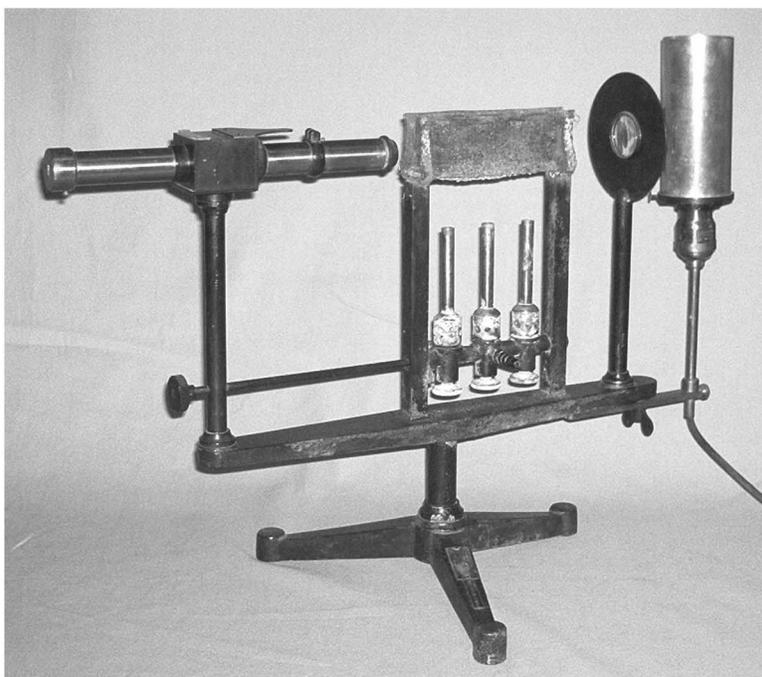
Sodium Absorption Apparatus. This apparatus is designed to show the black line spectrum produced when white light passes through sodium vapor. At the right is an incandescent light source with a narrow slit in the nickel-plated shield. The beam of light is focused on the entrance slit of the spectrometer at the left by the lens. Rock salt, placed in the middle section, is disassociated by the three gas burners. The exit side of the spectrometer swings from side to side to observe the spectrum and note the two black sodium D absorption lines in the otherwise continuous spectrum. The apparatus was made, ca. 1925, by the Eastern Scientific Company of Boston. This example is in the Greenslade Collection; another example is still used for demonstrations at Colgate University. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)