10-2017

Analysis and Observation of Moving Domain Fronts in a Ring of Coupled Electronic Self-Oscillators

Lars Q. English
Dickinson College

A. Zampetaki

P.G. Kevrekidis

Kevin Skowronski
Dickinson College

Christopher B. Fritz
Dickinson College

See next page for additional authors

Follow this and additional works at: https://scholar.dickinson.edu/faculty_publications

Part of the Physics Commons

Recommended Citation


This article is brought to you for free and open access by Dickinson Scholar. It has been accepted for inclusion by an authorized administrator. For more information, please contact scholar@dickinson.edu.
Authors
Lars Q. English, A. Zampetaki, P.G. Kevrekidis, Kevin Skowronski, Christopher B. Fritz, and Saidou Abdoulkary

This article is available at Dickinson Scholar: https://scholar.dickinson.edu/faculty_publications/785
Analysis and observation of moving domain fronts in a ring of coupled electronic self-oscillators

L. Q. English,1 A. Zampetaki,2 P. G. Kevrekidis,3 K. Skowronski,1 C. B. Fritz,1 and Saidou Abdoulkary4

1Department of Physics and Astronomy, Dickinson College, Carlisle, Pennsylvania 17013, USA
2Zentrum für Optische Quantentechnologien, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
3Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003, USA
4Département des Sciences Fondamentales, IMIP University of Maroua, P.O. Box 46, Maroua, Cameroon

(Received 30 April 2017; accepted 12 October 2017; published online 30 October 2017)

In this work, we consider a ring of coupled electronic (Wien-bridge) oscillators from a perspective combining modeling, simulation, and experimental observation. Following up on earlier work characterizing the pairwise interaction of Wien-bridge oscillators by Kuramoto–Sakaguchi phase dynamics, we develop a lattice model for a chain thereof, featuring an exponentially decaying spatial kernel. We find that for certain values of the Sakaguchi parameter $\alpha$, states of traveling phase-domain fronts involving the coexistence of two clearly separated regions of distinct dynamical behavior, can establish themselves in the ring lattice. Experiments and simulations show that stationary coexistence domains of synchronization only manifest themselves with the introduction of a local impurity; here an incoherent cluster of oscillators can arise reminiscent of the chimera states in a range of systems with homogeneous oscillators and suitable nonlocal interactions between them. Published by AIP Publishing. https://doi.org/10.1063/1.5009088

Mass-synchronization in biological systems has been studied extensively for several decades now using mathematical tools. The basic question is: how do individual, non-identical oscillators (neurons, fireflies, etc.) influence each other’s dynamics to give rise to group synchronization on the global scale. Yoshiki Kuramoto’s discovery that the synchronized state can arise via a second-order phase transition from the incoherent state led to a surge of interest in the field, as it opened up many new lines of investigation. What role, for instance, did the coupling topology play in the emergence of this temporal phase transition? More recently it was predicted that an intriguing hybrid state could stabilize itself in a lattice of identical oscillators, where a synchronized cluster could coexist with an incoherent cluster, and it soon began to be called the chimera state. While this state was initially solely a numerical and then an analytical prediction, more recently some experiments have verified its existence in a variety of systems. In many experimental contexts it is somewhat difficult to satisfy or even approximate the constraints imposed by the original theory, as it relates to the spatial coupling kernel, for instance, and some of the observations have also been indirect. Now there have been some theoretical reports of traveling domain fronts in non-locally coupled Kuramoto-type lattices. Here we report on direct observations of such traveling fronts, as well as impurity-induced chimera-like states in a lattice of 32 electrical self-oscillators. We also conduct a numerical exploration of the system’s approximate dynamics in good agreement with experimental observations. While our system does not appear to lend itself to true chimera states in the case of identical oscillators, it provides a promising example of nonlocal interactions in an electrical lattice that could be interesting to further explore (including in higher dimensional settings) as a rich dynamical example of pattern formation in a setting amenable to detailed spatio-temporal probing.

I. INTRODUCTION

Chimera states simultaneously featuring stable groups of synchronized and incoherent oscillators in coupled, spatially extended systems have attracted enormous attention ever since they were first seen more than a decade ago.1,2 This is likely due to the counter-intuitive symmetry breaking that such spatio-temporal patterns display. Kuramoto and Battogtokh first discovered the possibility of such symmetry-broken states within a network of identical phase oscillators in numerical simulations of the complex Ginzburg–Landau equation with non-local coupling.1 Soon thereafter, Strogatz and Abrams performed a rigorous analysis and also coined the term “chimera” for this novel state.2,3

In the years since this analysis, a diverse array of studies focusing on the numerical computation of chimeras revealed the wide applicability and robustness of such a feature (even against small spatial inhomogeneities) across many coupled-oscillator systems. Furthermore, distinct dynamical variants to Kuramoto’s original chimeras were discovered (transient vs. permanent, traveling vs. standing, etc.).4–6 The possibility of traveling fronts and bumps in such non-local and Kuramoto-coupled lattices has been investigated analytically and numerically.7,8 A classification of the different known chimera-like states can be found in the recent contribution of Ref. 9; see also references therein, as well as in Ref. 10, for a broad range of relevant examples.
On the experimental side, the number of available paradigms is far more limited. Nevertheless, recently chimera states have begun to be reported in mechanical, chemical, electro-chemical, and opto-electronic oscillator systems. An optical study also found chimera dynamics within liquid-crystal cells, where the coupling was accomplished via computer feedback. Similar patterns were also seen in an electronic FM oscillator with time-delayed feedback. In the mechanical case, a group of identical metronomes on elastically coupled swings revealed the possibility for a chimera state consisting of one synchronized group and another turbulent one. The observation of near-harmonic chimera states in which coherent and turbulent domains alternated in their location. A lattice of electronic oscillators mimicking neuron-like spiking was found to be divided into quiescent and synchronized domains in Ref. Furthermore, coupled Boolean phase oscillators were shown to exhibit transient chimeras with lifetimes that increased with the network size.

In this paper, we report direct experimental observations of traveling domain fronts involving one (or more) interval(s) of synchronized oscillators moving through anti-synchronized ones. We also observe local-impurity-induced quasi-chimera states (involving an incoherent cluster albeit for a non-perfectly homogeneous system) in a one-dimensional lattice of phase oscillators. Furthermore, we develop from first principles a corresponding theoretical model and perform numerical simulations of the equations approximately describing the experimental system and find a good qualitative agreement. The lattice consists of 32 Wien-Bridge oscillators, each one bi-directionally and resistively coupled to its two nearest neighbors in a ring formation. This system has been the subject of recent synchronization studies and shown to obey the Kuramoto–Sakaguchi phase oscillator model to a good approximation. Crucially, its dynamics can be fully captured (in a distributed, spatio-temporal way) and characterized experimentally.

While our observations do not reveal turbulent or chaotic states in the pure lattice, we do observe features that may be called “chimera-like” in the following generalized sense: our system supports traveling domains of nearly phase-locked oscillators inside of a standing, nearly out-of-phase background. These traveling fronts, seen both in the simulations and in experiments, are reminiscent of patterns recently discovered analytically in generic Kuramoto-like models. In both our numerical and experimental setting, stationary domains of synchrony or asynchrony could only be achieved by the introduction of a local impurity. Only in this case do we observe the formation of an incoherent cluster phase (not including the impurity site) whose coexistence with another synchronized or anti-synchronized phase would more typically be associated with a chimera. More broadly, both experimental and numerical results suggest that the existence of traveling fronts or chimera-like states relies on the presence of bi-directional coupling, as well as on the Sakaguchi–Kuramoto phase-delay parameter, \( \alpha \), set to a value within a fairly narrow window.

Our presentation will be structured as follows: we will start by describing the experimental setup and design (Sec. II), we will then present an approximate theoretical model emulating the relevant electronic system (Sec. III), in Sec. IV, we will give a series of simulations for guidance on the kind of phenomenology that the model may exhibit, and finally, in Sec. V we will present our experimental results, and in Sec. VI, we will summarize our findings and present some conclusions and future challenges.

II. EXPERIMENTAL SETUP

Figure 1 schematically depicts the experimental system. We couple 32 individual Wien-bridge oscillators in a ring topology. The oscillators have a natural frequency centered around 279.5 Hz with a standard deviation of 0.35 Hz. This means that the spread in natural frequency is only about 0.13% and hence for practical purposes the oscillators are nearly identical in this measure. The coupling between the nearest neighbors in the ring is accomplished resistively, and it can be either uni-directional or bi-directional in nature; shown in the figure is the latter configuration with two oscillators fleshed out in the top panel. All oscillators are

![FIG. 1. Schematic of the coupled oscillator ring circuit with two Wien-bridge oscillators shown. The coupling is accomplished via two resistors to each of the two neighboring oscillators in the ring: from the output of the oscillator’s op-amp to the non-inverting input of the two neighbors, via \( R_+ \), as well as to their inverting inputs, via \( R_- \). Thus, the output of each op-amp is connected to both inputs of the two neighboring op-amps via \( R_+ = R_- = 62 \, k\Omega \) resistors.](image-url)
monitored simultaneously by a 32-channel digitizer. In previous studies,\textsuperscript{20,21} only one resistor from the op-amp output of the first oscillator to the non-inverting input of the second (labeled $R_-$) was used to establish the (uni-directional) connection. The resulting oscillator phase interaction was found to be well described by a Kuramoto–Sakaguchi model with a fairly low value of the phase-delay parameter, $\alpha$. A second coupling channel between oscillators, via the resistor labeled $R_+$, now connects to the inverting input of the receiving op-amp, as shown in the diagram.

Since the effective value of $\alpha$ in the Kuramoto–Sakaguchi model plays an important role in the emergence of the chimera state,\textsuperscript{2} it is essential to determine and control it experimentally. For the purpose of measuring $\alpha$, two Wien-bridge oscillators were first tuned to the same natural frequency (to within achievable precision) before adding two resistors, $R_-$ and $R_+$, that couple them uni-directionally. This oscillator pair should then be described by the system,

$$\dot{\phi}_1 = \omega_1$$
$$\dot{\phi}_2 = \omega_1 + K \sin (\phi_1 - \phi_2 - \alpha).$$ (1)

If the two oscillators frequency-lock, $\dot{\phi}_1 = \dot{\phi}_2$, and this forces $\phi_1 - \phi_2$ to be equal to $\alpha$. Figure 2(a) shows this case for the symmetric coupling situation of $R_+ = R_- = 62 \text{k}\Omega$. As predicted, the driving oscillator (oscillator 1) leads the driven oscillator (oscillator 2) in phase. Measuring their phase difference then yields $\alpha$, as estimated by finding the zero-crossings for both traces, dividing their temporal difference by the period and multiplying by $2\pi$. It is also desirable to possess some experimental control of $\alpha$. Several modifications to the purely resistive coupling were tested for this purpose, such as incorporating coupling capacitors. However, simply adding the coupling resistor, $R_-$, proved to be the most efficient way to tune $\alpha$ over a large interval. The effect of this coupling resistor, $R_-$, is illustrated in Fig. 2(b). Note that here $R_-$ was fixed at 62 k\Omega. For very large values of $R_-$, the phase-delay parameter $\alpha$ is close to the value of 0.5 found previously, whereas for small values, $\alpha$ saturates at $\pi$. The most sensitive dependence is seen to occur within the resistance interval of 10 k\Omega and 100 k\Omega. In the subsequent data-sets, we choose $R_- = R_+ = 62 \text{k}\Omega$. This combination produces an $\alpha$-value of 1.2 rad.

III. THE MODEL

Under certain assumptions, the governing equations for the voltages at each node of the lattice can be derived as in Ref. 23, by starting from Kirchhoff’s node rule at the positive and negative Op-amp inputs of the $i$th oscillator. Relating the details to the Appendix, an intermediate result for the temporal evolution of the voltage at the $i$th node takes the following form:

$$V_i'' + \left(2 - \frac{R_1}{R_2} \frac{2R_1}{R_+} \right) V_i' + \left(1 + \frac{2R_1}{R_+} \right) V_i = \frac{R_-}{R_+} \left(V_i^\text{out}_{i+1} + V_i^\text{out}_{i-1} \right) \left(\left(V_i^\text{out}_{i+1}\right)' + \left(V_i^\text{out}_{i-1}\right)' \right) = 0. \tag{2}$$

Here, the prime denotes differentiation with respect to non-dimensional time $\tau = t/(RC)$. $R_1$ and $R_2$ are the two resistors of the inverting amplifier part of each Wien-bridge circuit, with $R_1$ taken to have a nonlinear voltage-dependence (due to the two diodes in parallel with it). The resistor $R_-$ is in the filter circuit from Op-amp output to positive (non-inverting) input, and $R_+$ and $R_-$ denote the two coupling resistors; for the details of these quantities in the coupled oscillator ring circuit, see Fig. 1.

Note that $V_i$ is the voltage of node $i$ at the input of the Op-amp, whereas $V_i^\text{out}$ is the voltage at the output of the Op-amp. Thus, Eq. (2) is not yet sufficient in order to solve for the voltages; we must also find a relationship between the input and output voltages. By examining current flow into or out of the node at voltage $V_i$ at the inverting input, we can derive the following relationship:

$$\left(1 + \frac{R_1}{R_2} + \frac{2R_1}{R_-} \right) V_i = V_i^\text{out}_{i+1} + \frac{R_1}{R_-} \left(V_i^\text{out}_{i+1} + V_i^\text{out}_{i-1} \right). \tag{3}$$

We can write this in the matrix form as,

$$V^\text{out} = c A^{-1} V^\text{in}, \tag{4}$$

where $c = 1 + \frac{R_1}{R_2} + \frac{2R_1}{R_-}$, and

$$A = \begin{bmatrix}
1 & \gamma & 0 & \cdots & \gamma \\
\gamma & 1 & \gamma & 0 & \cdots \\
0 & \gamma & 1 & \gamma & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\gamma & 0 & \cdots & \gamma & 1
\end{bmatrix}$$

with $\gamma = \frac{R_-}{R_+}$. $A$ is an $N \times N$ matrix encapsulating the novel nearest-neighbor coupling scheme. It is essentially a tridiagonal matrix but with two additional entries (in the top-
right and bottom-left corners) that enforce the periodic boundary conditions. If no coupling exists in the negative (inverting) input of the Op-amps, then \( \gamma = 0 \) and \( A \) reduces to the identity matrix.

Equation (4) can also be adapted to the derivatives of the voltages appearing in Eq. (2). Note, however, that in the experiment \( R_1 \) is voltage-dependent, and thus implicitly time-dependent. To make the modeling somewhat more tractable, let us consider the slightly modified Wien-bridge oscillator where the nonlinearity is contained in \( R_2 \), and where \( R_1 \) is constant. For concreteness, we assume that,

\[
R_2 = R_20(1 + bV_i^2).
\]

Given the symmetric nature of the diode configuration, we expect this voltage dependence to be generically valid for small voltages. Then, from Eq. (4) we obtain:

\[
V_j^{\text{out}} = c \sum_i A_{ij}^{-1} V_j - \frac{R_1}{R_2} \frac{dR_2}{dv_i} \left( \sum_j A_{ij}^{-1} V_j \right) V_i.
\]

Equations (4)–(6) together with Eq. (2) represent the governing equations of motion for the voltages on this lattice. A key observation is that the inverse of the tri-diagonal matrix \( A \) is not itself tri-diagonal, but rather involves a form of non-local coupling, although it will be heavily diagonal-centered for small \( \gamma \). Consequently, the system of Eq. (2) is, in fact, defined by lattice-node coupling that goes beyond the nearest-neighbor and hence bears a prototypical feature that is often associated with chimera-bearing states, namely, a non-local coupling. There exist analytic solutions for the inverse of symmetric tri-diagonal matrices like \( A \). In general, when the diagonal entries are all equal to \( a \) and the nearest off-diagonal entries equal to \( \beta \), then for \( i > j \), we have

\[
[A^{-1}]_{ij} = (-1)^{i+j+1} \frac{1}{\beta} \frac{U_{i-1}(a/2\beta)U_{n-i}(a/2\beta)}{U_n(a/2\beta)},
\]

where \( U \) are the Chebyshev polynomials of the form

\[
U_n(x) = \frac{\sinh(n + 1)x}{\sinh x}, \quad \text{and} \quad \cosh \theta = x.
\]

If we adapt this result to our matrix, we obtain the following formula for the entry in the first row of the inverse:

\[
[A^{-1}]_{1,1} = (-1)^{x+1} \frac{1}{\gamma} \frac{\sinh(N + 1 - x)}{\sinh(N + 1)},
\]

where \( x \) is an integer between 1 and \( N \), and \( \kappa = \cosh^{-1} \left( \frac{1}{\gamma} \right) \). Using the approximation that \( \sinh \kappa(N + 1) \approx \exp[\kappa(N + 1)]/2 \), which should hold very well for large lattices (of sufficiently large size \( N \)), we arrive at:

\[
[A^{-1}]_{1,1} \approx (-1)^{x+1} \frac{1}{\gamma} e^{-\kappa x}.
\]

Thus, we obtain a spatial kernel that is essentially exponentially decaying with a decay constant that depends only on \( \gamma \); the larger the value of \( \gamma \), the smaller is the decay constant. Note also the alternating sign in the kernel. To illustrate this point with a concrete example, assume that \( N = 100 \) and that \( \gamma = 0.1 \). This yields a decay constant of \( \kappa = 2.3 \), and the first row of \( A^{-1} \) reads:

\[
\{1.02, -0.103, 1.04 \times 10^{-2}, -1.05 \times 10^{-3}, 1.06 \times 10^{-4}, 1.07 \times 10^{-5}, \ldots, -0.103\}.
\]

This means that \( V_j^{\text{out}} \) will be predominantly given by \( V_i \), but \( V_0 \) and \( V_2 \) will still also play a role, with a progressively small contribution from \( V_{99} \) and \( V_3 \) and beyond. In the experimental system, we have \( R_1 = 27\,\text{k\Omega} \) and \( R_2 = 62\,\text{k\Omega} \), so \( \gamma = 0.435 \), which leads to a spatial decay constant \( \kappa = 0.54 \). This results in a weaker decay of the relevant kernel and a broader range inter-neighbor coupling than the example above. In any event, in our numerical computations that will follow, we consider all neighbors with the respective pairwise coupling as dictated by Eq. (7).

**IV. NUMERICAL SIMULATIONS**

In order to motivate the experimental results that will follow, we have performed numerical simulations of the aforementioned theoretical model, starting with random initial conditions (voltage values \( V_j(0) \) in the interval \([-1, 1] \) and \( V_j(0) = 0 \). By adjusting properly the values of the effective parameter \( b \) and the gain resistances \( R_1 \) and \( R_2 \), we can obtain stable oscillations in the long time dynamics over a large range of \( R \) values. Keeping \( R_1 \), fixed at 62\,\text{k\Omega} and incrementing \( R_2 \), we observe that the system of bi-directionally coupled oscillators passes from an anti-synchronized phase to a fully synchronized one.

In the intermediate regime, the dynamics of the oscillator ring is prone to the formation of chimera-like patterns consisting of mixtures of the two distinct phases (synchronized and anti-synchronized ones). As shown in Fig. 3, these typically have the form of traveling patterns of a nearly in-phase domain within a nearly anti-phase background, or vice versa. These traveling synchronization domains are seen to persist in time. The particular traveling chimera pattern can consist of a narrower [Fig. 3(a)], a wider [Fig. 3(b)] or even multiple [Figs. 3(a) and 3(c)] synchronized domains with both positive [Figs. 3(a) and 3(c)] and negative [Fig. 3(b)] velocities. Which pattern is finally realized depends on the exact parameters as well as on the initial conditions. Irrespective of the latter, however, the tendency of the model to form traveling chimera-like patterns in a specific parameter regime is clear. However, it should be highlighted that we label these states “chimera-like” (rather than genuine chimeras) because they bear the characteristic of co-existence of (in fact) moving domains of Wien-bridge oscillators, but neither of the distinct phases appears to be turbulent or incoherent in its nature within our numerical computations.

Given that our theoretical model favors the formation of traveling states (and not of stationary incoherent clusters), there is a question about whether the presence of an impurity can lock a number of oscillators in space. For this reason, we performed further numerical simulations in which the value of the resistance \( R \) of a certain oscillator differs from that of the rest in the ring. It is found that for a strong enough impurity (a difference of about 25%) a stationary pattern can be stabilized, as shown in Fig. 4. Here small intermittent
synchronized domains appear inside an anti-synchronized phase. Furthermore, the defect in this case can be classified, in accordance with the classification of [25], as a source (given its concurrent emission of wavetrains on the two sides).

V. EXPERIMENTAL RESULTS AND DISCUSSION

Motivated by the above computational illustrations of the potential for co-existing phases within the model, we now turn to the main core of the present contribution, consisting of experimental investigations of the potential formation of chimera-like states in this system. We once again caution the reader that the experimental system is only approximately described by the model, and hence the observed deviations from the simulation results may be attributed to model limitations. The main difference is (a) that we operate the experimental Wien-bridge oscillators at a higher gain value of \( g = 1 + R_{10} R_{20} = 10 \), (b) that the diode-pair is in fact lowering the effective resistance of \( R_1 \) and not increasing the resistance of \( R_2 \), and (c) that higher order terms may be needed in Eq. (5) to accurately model the effect of diodes. This latter feature is worthwhile of an additional independent study. Nonetheless, the model is phenomenologically consistent with the gain saturation for larger voltages caused by the diodes and already exhibits a number of intriguing phenomenological features motivating the presented experiments.

Having stated these caveats, let us turn to the experimental picture. We start the investigation with near-identical oscillators coupled in the uni-directional ring topology. Here, the oscillators synchronize themselves into a perfectly phase-locked mode that obeys the periodic boundary condition. No oscillators are drifting in this state.

FIG. 3. Three different cases (a), (b), and (c) of the simulation results obtained by the aforementioned theoretical model [Eqs. (4)–(6) together with Eq. (2)] exhibiting a chimera-like behavior. Displayed are the simulated voltages (depicted by the color) as a function of time (x-axis) and oscillator index (y-axis). The left panels account for the dynamics of the system in a large time interval whereas the right panels are zooms of the long-time behavior. Obviously there is a traveling chimera-like in all three cases with a different width and number of the synchronized domains. In all cases, \( R_+ = 62 \, k\Omega \), \( R_1 = 9 \, k\Omega \), \( R_2 = 2.7 \, k\Omega \), and \( b = 5 \), whereas for (a) \( R_- = 100 \, k\Omega \), \( R_+ = 4.7 \, k\Omega \), (b) \( R_- = 118 \, k\Omega \), \( R_+ = 4.7 \, k\Omega \), and (c) \( R_- = 118 \, k\Omega \), \( R_+ = 4 \, k\Omega \).

FIG. 4. Simulated voltages (displayed as color) as a function of time (x-axis) and oscillator index (y-axis) for the case of Fig. 3(a) but with an impurity. The impurity is due to the value of the resistance for the 15th oscillator being \( R = 3.5 \, k\Omega \), whereas for all other oscillators \( R = 4.7 \, k\Omega \).

FIG. 5. The measured voltages (displayed as color) as a function of time (x-axis) and oscillator index/position (y-axis). The connections are all uni-directional (clockwise around the ring). The oscillators quickly synchronize into a perfectly phase-locked mode that obeys the periodic boundary condition of \( \Delta \phi = 2\pi m / N \). This is illustrated in Fig. 5, which depicts a mode number of \( m = 9 \), resulting in a phase advance of \( \Delta \phi = 1.76 \, \text{rad} \). Here, the coupling resistors, \( R_- \).
and $R_+$, are both at 62 kΩ. Such forward-facing rolls of synchronization are very stable states for this coupling configuration. Since each oscillator is effectively driven by the previous one without acting back on it, such patterns are naturally expected.

When additional coupling resistors of the same value (62 kΩ) are added to make the ring bi-directional, however, the dynamics changes significantly. In this situation, due to the underlying symmetry one might expect a state of perfect synchrony or anti-synchrony, corresponding to a mode number of $m=0$, and this is indeed sometimes observed. Figure 6 depicts a typical data-set. The upper panel, Fig. 6(a), reveals a checkered pattern of synchronization, where all pairs of neighboring oscillators are exactly out-of-phase. This state competes with the more dynamically interesting one shown in Fig. 6(b). Here, we observe that fronts comprised of domains of nearly in-phase oscillators travel through the checkered state at a well-defined speed. Experimentally, these fronts, here three in number, persist indefinitely once they have established themselves. One way to generate them consistently from the ambient checkered pattern is to briefly introduce a temporary impurity somewhere in the lattice (by detuning a resistor) and then to quickly restore the periodicity of the lattice. This experimental result is very reminiscent of the numerical simulations of Fig. 3.

In order to investigate these traveling fronts of Fig. 6(b) more closely, let us examine the dynamic oscillator frequencies. A straightforward way of computing these frequencies uses the interpolated (upward) zero-crossings in the voltage time-series (as described in more detail in Ref. 21), and this produces discrete frequency values that are averaged over each period of oscillation. Figure 7(a) plots these dynamic frequencies computed from the raw data in Fig. 6 for oscillator $x=1$. We see that in the checkered pattern an oscillator will have a frequency of around 360 Hz, but this periodically decreases to around 270 Hz at those times when it momentarily becomes a part of the traveling front.

We can also extract the phase information of the individual oscillators from their raw voltage data. This was accomplished using the well-known Hilbert transform (described, for instance, in Ref. 20). Note that the phases obtained in this fashion do not advance at exactly constant speeds over a single period, as the voltage oscillations deviate substantially from sinusoids. Nonetheless, inspection of the phase-profiles indicates no abrupt or large-scale fluctuations in phase-velocity within a single period of oscillation, thus providing us with a useful measure of the oscillator phase. Figure 7(b) shows the phase of each oscillator at a particular instance of time, $t=0.05 \text{s}$. Here, we observe that the nearest-neighbors are mostly $\pi$ out-of-phase, characteristic of the checkered phase. However, the fronts reduce that phase difference substantially between a group of three neighboring oscillators.

The simulations revealed that upon the introduction of a local impurity a chimera-like state (involving an intermittent cluster of stationary oscillators) could establish itself; see Fig. 4. A similar phenomenon is also observed in the experiments, as illustrated in Fig. 8(a). Here, we introduce an impurity via the detuning of the resistor, $R$, at node $x=19$. The measured voltages (displayed as color) as a function of time (x-axis) and oscillator index/position (y-axis). Here, the connections are bi-directional. (a) The anti-synchronized state and (b) traveling fronts of synchronized domains can be seen to propagate within the anti-synchronized state. Whether (a) or (b) is observed depends on the initial conditions.
between oscillators, so that a resistor controlling $R$ by roughly 15%, a very different oscillator organization is observed. The symmetry-breaking now stimulates the appearance of two sets of rolls, but in the region where they meet in the ring lattice, we observe a state reminiscent of a chimera, where some oscillators’ phases are incoherently relative to the synchronized clusters. (a) The experimental oscillator voltages as a color density plot. The color bar is the same as the previous. (b) The frequencies as a function of time of four oscillators near the incoherent cluster, as computed from the experimental data. Oscillator $x = 10$ is quite stable in the frequency, at around 260 Hz, as is oscillator $x = 5$ at the center of the cluster. However, oscillators near the edges of the cluster at $x = 4$ and 8 (dotted and dot-dashed lines) perform wide frequency swings.

by around 15%–20%. The result is that we get a kink in the roll pattern at that location, i.e., a source pattern similar to the simulations. However, the pattern remains locally phase-coherent. Only at the other side of the ring-lattice, between nodes $x = 3$ and 8, do we now see a different phase pattern take root. At the boundaries of this domain, a few of the oscillators do not maintain a fixed phase relationship with either domains, and may be considered incoherent, and this is likely due to competing interactions induced by the periodic boundary conditions and the wavetrains emanated at the defect site. In Fig. 8(b), we plot the dynamic frequencies of four oscillators near the incoherent cluster. Oscillator $x = 10$ is quite stable in frequency, at around 260 Hz, as is oscillator $x = 5$ at the center of the cluster. However, oscillators near the edges of the cluster at $x = 4$ and 8 (dotted and dot-dashed lines) perform wide frequency swings, as they try to bridge the two separate phases.

To demonstrate that the traveling fronts and chimera-like states rely on a sufficiently large value of the phase-delay parameter, $\chi$, we can remove all $R_z$ coupling resistors between oscillators, so that $\chi$ reverts back to the lower value of around 0.5 rad. The $R_z$ resistors are still in place to provide bi-directional coupling. All oscillators are then phase-locked again with one another in a pattern of near vertical rolls. No oscillators are observed to drift in phase, and even if this pattern is temporarily perturbed via a momentary introduction of an impurity, it quickly reconstructs itself when the impurity is gone. In this $\chi$ regime, a chimera-like state cannot be induced in the system.

In summary, Ref. 9 recently proposed a classification scheme for chimeras resting on spatial and temporal correlation parameters and their distributions. According to this scheme, we see stationary moving states. The discrete second spatial derivative is approximately zero in the sync- and anti-sync domains, but at their boundaries, it is non-zero. With a local impurity, however, we also observe something akin to stationary static chimera-like waveforms (in accordance with the diagnostics and the classification developed therein).

VI. CONCLUSIONS AND FUTURE WORK

We have found both experimentally and numerically that the ring-lattice of bi-directionally coupled Wien-bridge oscillators with its exponentially decaying spatial kernel admits moving domain fronts of synchronized oscillators within a background of anti-synchrony. A chimera-like state could also be observed as a result of introducing a source defect/impurity within a periodic chain (presumably as a result of the stationary, intermittent cluster resulting from the interaction of the two counter-propagating wavetrains emanating from the defect). On the contrary, uni-directionally coupled oscillators feature uniformly phase-locked states. We motivated the co-existence of different phases through a theoretical model and its numerical simulation; in fact, the latter featured the traveling of one or even multiple intervals of oscillators of one type (in-phase) within a pattern of a different type (anti-phase). The experimental measurements of a ring of 32 oscillators that followed this qualitative theoretical/numerical investigation constitute an intriguing paradigm of a non-locally coupled electrical oscillator system that may be relevant to further explore rich, spatio-temporally resolvable pattern-forming dynamics. In the experiments presented herein, we found evidence of the traveling synchronization domains, as well as of the intermittent stationary clusters in the presence of the defect in a good qualitative agreement with our numerical findings.

Naturally, the present work opens a significant array of new directions. On the one hand, while here we examined some prototypical parametric regimes of the model, it would be of interest to conduct a deeper theoretical study of the possible outcomes of the numerical experiments. On the other hand, it would be especially useful to attempt to systematically characterize the dependence within Eq. (5) of the resistance on the voltage (i.e., the source of the model’s nonlinearity) with the aim to produce a model that more adequately captures the experimentally observed phenomenology. Armed with these theoretical/numerical tools, one could also embark in other directions associated with the potential that this type of highly tractable, space-time resolved experiments offer. In particular, in line with some of the most recent theoretical investigations, one could explore the possibility of formation of chimera-like states in higher dimensional setups, such as the twisted and spiral chimera states of Ref. 26. Such studies will be deferred to future investigations.
ACKNOWLEDGMENTS

We thank David Mertens for some helpful early discussions about oscillator ring lattices. P.G.K. gratefully acknowledges the support of NSF-PHY-1602994, as well as from the ERC under FP7, the Marie Curie Actions, People, International Research Staff Exchange Scheme (IRSES-605096), the Alexander von Humboldt Foundation, and the Stavros Niarchos Foundation via the Greek Diaspora Fellowship Program. We thank the reviewers for their valuable feedback.

APPENDIX: DERIVATION OF THE THEORETICAL MODEL

Here, we give some of the steps that lead from the circuit equations to the system’s governing equations of Eq. (2). We start by considering the currents going in and out of a node at the negative op-amp input of the $i$th oscillator. By the Kirchhoff node rule in the circuit of Fig. 1

\[ \frac{V_i}{R_2} = \frac{V_{out} - V_i}{R_1} + \frac{V_{out}^{i+1} - V_i}{R_-} + \frac{V_{out}^{i-1} - V_i}{R_-}. \] (A1)

This then leads to

\[ \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{2}{R_-} \right) V_i = \frac{V_{out}^{i+1} + V_{out}^{i-1}}{R_1}. \] (A2)

Now, if we examine the positive op-amp input, we see that

\[ V_{out}^i - V_i = V_{cap} + I_1 R, \] (A3)

where $I_1$ is the current flowing through resistor $R$ in series with capacitor $C$ and $V_{cap}$ is the voltage across the capacitor. However, the current $I_1$ can be computed as follows:

\[ I_1 = C \dot{V}_i + \frac{V_i}{R} + \frac{1}{R_-} (V_i - V_{out}^{i+1} + V_i - V_{out}^{i-1}). \] (A4)

Substituting Eq. (A4) into Eq. (A3) yields

\[ V_{out}^i = V_{cap} + \frac{(RC)}{R} \dot{V}_i + \frac{R}{R_-} \left( 1 + \frac{R}{R_+} \right) V_i - \frac{R}{R_+} \left( V_{out}^{i+1} + V_{out}^{i-1} \right). \] (A5)

Combining Eqs. (A2) and (A5), we take another time derivative on both sides, substitute $C \dot{V}_{cap} = I_1$, and obtain after some further manipulation the following form:

\[ RC \ddot{V}_i + \left( \frac{2}{R_1} - \frac{2}{R_2} + \frac{2}{R_-} \right) \ddot{V}_i + \left( \frac{1}{RC} + \frac{2}{R_-} \right) V_i - \frac{1}{R_-} \left( V_{out}^{i+1} + V_{out}^{i-1} \right) = 0. \] (A6)

Finally, introducing non-dimensional time, $\tau = \frac{t}{RC}$, we arrive at Eq. (2).

8C. R. Laing, Chaos 26, 094802 (2016).