5-17-2015

Stability and ILM Formation on a One-Dimensional Lattice

Marc Julien Besson
Dickinson College

Stability and ILM Formation on a One-Dimensional Lattice

Submitted in partial fulfillment of honors requirements
for the Department of Physics and Astronomy, Dickinson College,

by

Marc Besson

Advisor: Professor Lars English

Reader: Professor David Jackson
Reader: Professor Catrina Hamilton-Drager
Reader: Professor David Mertens
Reader: Professor David Reed
Carlisle, PA
May 11, 2015

Abstract

Spin waves have been researched in quantum physics and solid state physics since the discovery of spin wave solutions to a magnetic lattice by Felix Bloch in the early 20th century. In this thesis, we analyze a one-dimensional ferromagnetic lattice from a theoretical standpoint. In particular, we find the dispersion relation for spin waves in a one-dimensional anisotropic lattice, and determine the stability of this solution with respect to modular perturbations. This thesis aims to further establish the pathways and conditions under which spin wave solutions are unstable and lead to localized mode solutions.

Acknowledgements

I would first like to thank all the people that have made this possible. In particular, I thank Professor English for his time, advice, encouragement and mentoring. I also thank Professor Pearson for critiques and help. I would further like to thank Ben Kimock, my research partner, as well as the rest of the 491/492 Physics class of 2015.

Contents

List of Figures

Chapter 1

Introduction

Spin waves, also known as magnons, are the quantum solutions that describe mechanics on a magnetic lattice, and they have been studied since the early era of quantum mechanics. In particular, the first paper exploring the fundamentals and theory of spin waves is due to Felix Bloch in 1930 [1]. His work on determining the possible energy eigenstates on Bravais lattices based on translation operators is the foundation of the theory of spin waves. The theory of magnons and spin waves (we shall use the term spin waves exclusively from this point forward) was greatly expanded upon by Freeman Dyson in the 1950's [2], and were first experimentally observed by Brockhouse in 1957. Spin waves continue to be studied from the theoretical, numerical and experimental standpoint today.

Spin waves are propagating disturbances in the spins of ions in a lattice, such as a ferromagnet, a ferrimagnet or an antiferromagnet. Their existence has been experimentally confirmed on all three of these differing materials [3]. For instance, in the most basic example of the ferromagnet, certain regions of the material are ordered in a crystal lattice such that the magnetic moment of every ion in the lattice points in the same direction (known as the easy axis). Perturbations of this perfect ordering (which can arise in a number of ways, the most common being thermal energy) result

in waves which propagate along the lattice, due to the interactions of neighboring magnetic moments.

The fundamentals of spin waves are nearly eight decades old, but it remains an open field of study. There are several labs worldwide which conduct research, mostly in exciting and detecting spin waves, such as the Max Planck Institute in Halle, L'Institut Laue-Langevin in Grenoble and the National Institute of Standards and Technology in Gaithersburg. Understanding the mechanics of spin waves is important to understanding magnetism in solid state physics at the atomic level; given a ferromagnet or an antiferromagnet, regions or domains of the crystal have common orientation and order. As energy is increased (usually through temperature), this structure and organization begins to break down. Curie temperature or Neél temperature, for ferromagnets or antiferromagnets respectively, is defined as the temperature at which the spontaneous magnetisation resulting from an ordered lattice is destroyed; above these temperatures the thermal energy in the system is great enough that the permanent magnetization of the lattice gives way to induced magnetization. Thus, above this temperature (which varies from material to material), the material becomes paramagnetic. Spin waves are essential for our understanding of the mechanics of the energy in the system. Furthermore, spin waves are thought to be an important source of energy loss when solid state crystals such as ferromagnets are used in high energy systems [4].

This particular project will expand upon earlier research, especially [5] and [6], which deal with simulating spin waves and intrinsic localized modes (ILMs) on antiferromagnetic lattices. This earlier research presents a thorough treatment of spin waves in antiferromagnets, and proposes a new theoretical approach to the Hamiltonian of the system in one-dimensional crystal lattices. This project will seek to expand upon this result, examining the stability and ILM formation on 1-Dimensional ferromagnetic lattices. The principal features of the Hamiltonian we use are the exchange field

between nearest-neighbor particles and the anisotropy field. Furthermore, this thesis will attempt to determine the role of spin waves in the generation of ILMs, or intrinsic localized modes, which are local concentrations of energy on the lattice. This thesis is almost entirely theoretical, and has no experimental component.

Chapter 2

Theory

2.1 Lattices

In quantum mechanics, observables are represented as operators acting on an n-dimensional vector space, or Hilbert space, and a physical state is represented by a state vector, or a ket, $|v>$ [7]. Observables act on the state via matrix operation on vectors. In general, before a measurement is taken of a system with respect to a particular observable, the system's state is a normalized linear combination of the eigenvectors of the observable, known as eigenkets. When the observation is performed, the system is collapsed into one of the eigenkets, or eigenstates. Thus, the set of eigenkets and eigenvalues associated with those eigenkets provide us with a categorization of the system under observation. In the following section, we show that spin waves are the eigenstates for Bravais Lattices.

Spin waves can often be viewed from a classical perspective, but the foundations and initial derivations must be done from a purely quantum mechanical standpoint. Given a Hamiltonian H for our magnetic lattice, we wish to find eigenstates for energy. Thus we are searching for kets, or state vectors, that satisfy

$$H|v>=\epsilon_0|v>, \tag{2.1}$$

where ϵ_0 is a complex number. For systems with boundary conditions, or when we do not permit the lattice to be infinite, the derivation shows that ϵ_0 must be real. In this thesis we will only treat infinite lattices, so we simply treat ϵ_0 as complex to keep full generality.

We shall assume we are working with a Bravais lattice, which is an abstract model for physical crystals. A Bravais lattice is an infinite set of discrete points in 3-dimensional space, defined by a vector $\mathbf{R}$ known as the lattice vector,

$$\mathbf{R} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3}, \tag{2.2}$$

where each of the $\mathbf{v_i}$'s are primitive vectors denoting translational symmetries in the lattice, and the a_i's are the appropriate coefficients. For instance, the lattice vector $\mathbf{R} = \hat{x} + \hat{y} + \hat{z}$ would denote the Bravais lattice with a node or point at every integer triplet (a, b, c) in euclidean space $\mathbb{R}^3$. The set $\{\mathbf{v_i}\}$ spans the space of points, so we can write any lattice point as a linear combination $n_1a_1\mathbf{v_1} + n_2a_2\mathbf{v_2} + n_3a_3\mathbf{v_3}$, where $n_i \in \mathbb{R}$. This formula is required for full generality in a lattice, but in this thesis we will work with an optimal lattice such that $a_i = 1$; that is, the lattice spacing is uniform along all the primitive vectors. Thus, if we "shift" an infinite lattice by a_1 in the direction of v_1, or by any integer multiple of a_1, the lattice will remain invariant under that transformation. We would denote a shift by one unit in the $\hat{x}$ direction by $T(\hat{x})$, and for the euclidean lattice, $T(\hat{x})(x, y, z) = (x + 1, y, z)$. This is called the translation operator.

If we make the assumption that the Bravais lattice we are working with is infinite, then we know that it is translationally invariant so long as we translate the entire lattice by a linear combination of the primitive vectors with integer coefficients. Thus,

for the euclidean lattice, any translation $T(a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}})$ where $a_i \in \mathbb{Z}$ will leave the lattice invariant.

Thus, given the translation operator $T(\mathbf{R})$ for a given lattice vector, and any function defined on position $f(\mathbf{r})$,

$$T(\mathbf{R})f(\mathbf{r}) = f(\mathbf{R} + \mathbf{r}). \tag{2.3}$$

This has the simple effect of evaluating the function at $\mathbf{R} + \mathbf{r}$ instead of the original point, $\mathbf{r}$.

As we will see in the next section, since the Hamiltonian depends on potential, it thus is also periodic with respect to the lattice. For instance, the first term of the Hamiltonian we use in this thesis gives an expression for energy at lattice point n in a one-dimensional lattice that simply depends on the spins at lattice points $n-1$ and $n+1$, and this is clearly periodic with the lattice points. If the Hamiltonian is periodic with respect to the lattice vectors, it must remain invariant under the translation operator;

$$T(\mathbf{R})H(\mathbf{r}) = H(\mathbf{R} + \mathbf{r}) = H(\mathbf{r}). \tag{2.4}$$

We can use this to show that a Hamiltonian which is periodic with respect to the lattice commutes with the translation operator. If $\psi(\mathbf{r})$ is the wavefunction defining the state of the system, we have

$$T(\mathbf{R})H(\mathbf{r})\psi(\mathbf{r}) = H(\mathbf{r} + \mathbf{R})\psi(\mathbf{r} + \mathbf{R}) = H(\mathbf{r})\psi(\mathbf{r} + \mathbf{R}) = H(\mathbf{r})T(\mathbf{R})\psi(\mathbf{r}). \tag{2.5}$$

Since $T(\mathbf{R})$ and H commute, the eigenstates of H can be chosen to be simultaneous eigenstates of $T(\mathbf{R})$ for all $\mathbf{R}$ [8]. Note that we have kept these derivations as general as possible with regards to the Hamiltonian; so long as we have a translationally

invariant Hamiltonian, these considerations hold. So by finding eigenstates of H that are also eigenstates of $T(\mathbf{R})$, we will be finding an ansatz for the state of the system that we will use given a particular H as defined in [5].

The following result, known as Bloch's theorem, provides us with the general form of eigenstates of both a periodic Hamiltonian and the translation operator;

$$T(\mathbf{R})\psi = e^{i\mathbf{k}\cdot\mathbf{R}}\psi \tag{2.6}$$

with wave vector $\mathbf{k}$. The derivation of this theorem can be found in [9], and different derivation can be found in [7].

Thus, the general form of the eigenstates is that of a plane wave, and the wave function at different points will pick up a phase factor which is dependent on the wavelength. This is our general spin wave solution to motion on a lattice.

It is crucial to note that very few assumptions have been made as to the nature of the Hamiltonian H, other than the fact that it is periodic with respect to the lattice vectors. Thus, so long as the Hamiltonian we use is periodic over a lattice and we are working with mechanics on that lattice, we know the general form of the energy eigenstates.

2.2 Spin Waves

There are many crystals which are not magnetic, because frequently every electronin a shell is paired with another electron, and their spins cancel, resulting in no net magnetic moment for the ion. Ferromagnetic materials are crystals with a rigid lattice structure, such that there is an ion at every lattice point. Furthermore, each of these ions have unpaired electrons, resulting in a net magnetic moment. Ferromagnets have the property that below a certain critical temperature, known as Curie temperature, macroscopic regions (also known as domains) of the crystal have a preferred alignment

for the magnetic moments of the ions in the lattice. That is, over macroscopic domains of the lattice, there is a uniform magnetization in one direction, such that all of the ions in the domain have magnetic moments oriented in the same direction. We will refer to this axis of preferred common orientation as the easy axis.

Throughout this paper, the z-axis will denote the easy-axis orientation, and so the component of spin oriented along the easy-axis at lattice point i can be denoted as s_i^z. The lowest possible energy state is to have the entire crystal be one large domain, with every magnetic moment oriented along the preferred orientation (along the easy axis). As energy is added, usually in the form of heat, this ordering breaks down, and at the Curie temperature the magnetic spins are no longer correlated to each other, leading to a breakdown of the macroscopic magnetic field [8].

If we examine an ideal one-dimensional ferromagnetic lattice, the lowest energy state, which we denote $|0>$, will be that in which every single magnetic spin (and associated magnetic moment) is oriented along the easy-axis, as expected. Thus $s_n^z|0>= s_n|0>$, where s_n denotes the spin of the n^{th} lattice point. One could postulate that the first excited state $|1>$ would be where one of the magnetic moments (without loss of generality, the n^{th} lattice point) was pointed away from the easy-axis. In mathematical formalism, we would denote this

$$s_n^z|1>= (s-1)|1> . \tag{2.7}$$

While this is the correct idea, this particular state is not an eigenstate of the system. Since we are on a lattice and there is not an infinite energy potential between the lattice sites, our eigenstate must be invariant under the translation operators. Under the translation operators, there is qualitatively no difference between having the first spin be excited, or the second spin excited (because we can translate these states), or the k^{th} spin excited. The k^{th} magnetic moment is also linked to the neighboring lattice points through an exchange field, so there is a physical mechanism

through which the excited spin can propagate throughout the lattice.

In the most simple model, each of the lattice ions has spin $\frac{1}{2}$, and while real systems may be more complicated, this thesis only treats spin-$\frac{1}{2}$ lattice points. Thus, the first excited state is represented as a normalized superposition of states where the k^{th} lattice point has spin $s_k^z|1>=(s-1)|1>$, for all k. While an observation in time would localize the excited spin at a particular lattice point, in general we must look at the expectation value of the spins. The expectation value of the component of ion spin oriented along the easy-axis takes on continuous values from -1 to 1, based on how much energy is in the lattice and how many lattice points are in the system. Note that the situation of the expectation values of the spins is entirely different from the situation in which we actually measure the spins along some axis; in the latter case, there are only two options, $+\frac{1}{2}$ or $-\frac{1}{2}$. In this thesis we work with expectation values, so henceforth when we refer to "spin" or a magnetic moment, we refer to the expectation values, which take continuous values from 1 to -1, and not the measured quantities, which can only be $+\frac{1}{2}$ or $-\frac{1}{2}$.

The inherent interest lies in the component of the magnetic moment oriented towards the easy-axis and the component of declination away from the easy-axis. With this in mind, we introduce circularly polarized coordinates. From the observables s_n^z, s_n^x and s_n^y, we can form the expression

$$s_n^+ = \frac{s_n^x + is_n^y}{|s_n|}. \tag{2.8}$$

Here the magnitude of s_k^+ can be interpreted as the amount of spin orthogonal to the easy axis, or the declination of the spin away from the easy-axis, and s_k^+ is the combination of this magnitude and a phase in the plane. This value is a combination of the spin components in the x and y axes, and has been normalized. We can combine this interpretation with 2.6 to find that the natural picture of a spin wave in a one dimensional lattice is where every magnetic moment has been displaced from

equilibrium (aligned slightly away from the easy-axis) by a certain amount. Since magnetic moments in the presence of a magnetic field will precess, in the presence of an external or macroscopic magnetic field in the easy-axis direction, the magnetic moments will precess around the easy-axis.

With all this information, we can present the spin wave ansatz used in this thesis;

$$s_n^+ = f e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{2.9}$$

$$s_n^z = \sqrt{1-f^2}, \tag{2.10}$$

where n is the lattice point, r is the vector to the point, k is the wavevector, ω is the angular frequency, t is time, and f is a constant.

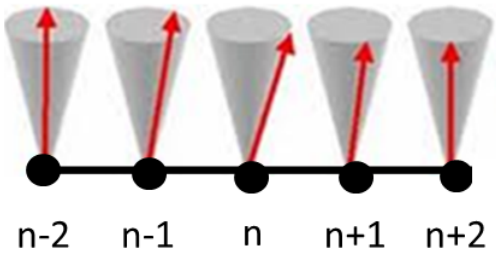


Figure 2.1: A segment of the one-dimensional lattice, with the expectation values of spin differing at each lattice point.

Note in particular that for the spin wave solution, while s_i^+ changes in time for all lattice points i, the z-component of spin s_i^z is constant in time.

2.3 Hamiltonian

Thus far the only assumption about the Hamiltonian of the system was that it was periodic with respect to the lattice. Since spin waves are eigenstates of the Hamiltonian, the the particulars of the spin wave depend on the Hamiltonian of the system

they are in. In this thesis, the 1-Dimensional Hamiltonian we will be working with is

$$H = -2J \sum_n \mathbf{s}_n \cdot \mathbf{s}_{n+1} - D \sum_n (\mathbf{s}_n^z)^2. \tag{2.11}$$

The two terms are as follows: the first term, known as the Heisenberg term or the Nearest Neighbor term, provides us with information on how the magnetic moment of ion n responds to the magnetic moments of ions $n-1$ and $n+1$ in a 1-dimensional lattice. Here J is a coupling constant, pertaining to the strenght of this nearest-neighbor interaction. This particular term being negative gives the model for ferromagnetism, but antiferromagnetism is handled when this term is positive.

The next term is the anisotropy term,

$$-D \sum_n (\mathbf{s}_n^z)^2. \tag{2.12}$$

This term depends on every element in the 1-Dimensional lattice. In essence, this term quantifies the fact that in a ferromagnet or an antiferromagnet, below the Curie temperature or the Neèl temperature, has one or more "preferred" axes along which the spins align. These preferred axes are determined by the symmetries of the crystal involved. Thus, even in the absence of an external applied magnetic field, there is a preferred orientation, which will act on the magnetic moment of a given ion, causing a change in its motion (if it is not aligned with one of the easy axes). This anisotropy term defines the easy axis as the z-axis, with D being the anisotropy constant for the given crystal. Note that while D depends not only on the crystaline structure but also the temperature, we do not include a temperature dependence in our calculations (we merely suppose that we are sufficiently cold so that thermal disruption to the anisotropy does not take place).

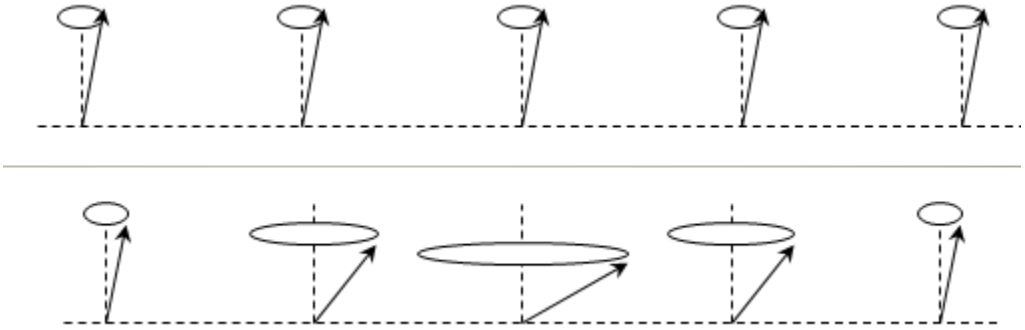
2.4 ILMS

Intrinsic Localized Modes, or ILMs, (sometimes also known as lattice solitons) are solutions to nonlinear, dispersive differential equations which are fundamentally different from the general wave solutions. For instance, the Sine-Gordon equation and the Korteweg-de Vries equation are both differential equations used to model certain phenomena which accept not only wave solutions but also soliton, or ILM solutions on lattice points. These two examples are notable in that the soliton solutions can be found analytically.

An ILM or soliton is a "solitary" wave, or a localized oscillation. Though oscillatory in nature, ILMs are distinct and separate from spin waves, which are true wave solutions. This thesis concerns itself with the stability of spin waves on a one-dimensional lattice; that is, under what conditions does the original spin wave solution break down. When the motion can no longer be described by the spin wave equations, we have in general an unstable system. If this system has an oscillatory localization of energy, we call it an ILM. In one dimension, this instability for antiferromagnetic lattices has been explored in [6] and in [5]. This thesis will deal with the stability of spin waves on a ferromagnetic one-dimensional lattice.

In particular, this thesis will consider modular perturbations. That is, given a spin wave solution to the Hamiltonian, we will superimpose a "perturbation" in the form of an oscillation with small amplitude compared to the amplitude of the spin wave. We then determine how the amplitude of the spin wave changes in time. To be specific, if the perturbation function grows in amplitude, our system is unstable with respect to that type of oscillation. If the perturbation decays away, we have a system which is stable with respect to that perturbation. If the amplitude of the perturbation function is unchanged, we have a system which is meta-stable with respect to the perturbation. Note that modular, or oscillatory perturbations, while representing a large class of perturbations, are not exhaustive. That is, even if a

Figure 2.2: Spin Wave Vs. ILM. The top image represents a possible spin wave, and the bottom represents one of many different possible ILMs. Note that with the ILM, the component of spin along the z-axis is not the same at every lattice point; thus, there is a concentration of energy at certain lattice points.



system is stable with respect to modular perturbations, it may still be sensitive to a variety of other perturbations described by different funcitons. Furthermore, there are other perturbations, such as randomized fluctuations in spin amplitude, which is the best model for heat. Modular perturbations were chosen because they are easy to work with analytically, and also cover a large number of possible perturbations.

2.5 Dispersion Relations

The main goal of this thesis is to analyze the stability of spin waves on a 1-Dimensional lattice, and to understand at what points they are unstable and when ILMs may form. To do this, we will need to understand the dispersion relation of spin waves. A dispersion relation is an equation which describes how a medium affects a wave passing through it. In general, dispersion occurs when oscillations of different frequencies propagate differently through a medium, which in our case is a lattice. So our dispersion relations will relate the wavenumber k of the spin wave with the frequency ω of the spin wave. The use of this relation will be key in the stability analysis of the system, so one of the first derivations will be the dispersion relation for a spin wave

on a 1-Dimensional ferromagnetic lattice.

2.6 Uniform Mode

The general equation of a wave is $fe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, where f is the amplitude, ω is the temporal frequency of the wave, $\mathbf{k}$ is the wavevector (which can be thought of as spatial frequency) and $\mathbf{r}$ and t represent position and time, respectively. This is the most general form of the wave, but when we are considering waves on a lattice, the behaviour is different due to the discrete nature of points at which the wave can "exist". Thus for instance, the spin wave can only "exist" at points of the lattice, so in our spin wave equation, $\mathbf{r}$ will always be a lattice vector in the sense discussed above. With these ideas in mind, it will prove useful to examine the "uniform mode", or the wave with wavevector $\mathbf{k} = 0$. This refers to the wave solution in which the magnetic moment at every lattice point is the same.

Figure 2.3: Uniform mode, and an example of a different spin wave. This diagram shows the planar component of the magnetic moment at a certain time t. For the uniform mode, the spins always remain in phase as time progresses. Other spin waves have a phase offset between different lattice points which persists as time progresses.

The uniform mode is of interest to us because generally, calculations with the uniform mode are much more simple since we set $\mathbf{k} = 0$. Furthermore, previous research by [5] indicates that in antiferromagnets, the uniform mode is particularly unstable. For these two reasons, we will attempt to find stability results for spin waves in general and also pay particular attention to the uniform mode.

Chapter 3

Derivation of Dispersion Relation

The two equations of utmost importance that we use in this thesis are the spin wave equation, given by

$$s_n^+ = f e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{3.1}$$

$$s_n^z = \sqrt{1-f^2}, \tag{3.2}$$

and the Hamiltonian. The interaction of these two equations, as well as the perturbation equation, is the focus of this thesis. Note that we keep $\mathbf{r}$ a vector, and use a dot product. While this is unnecessary in the one-dimensional case, it is crucial to have this distinction if this research is extended to two or three dimensions. Thus we keep the notation that would apply in other dimensions. Having already introduced both of these equations, we now begin the analysis, first by determining the dispersion relation imposed on the spin wave solution by our Hamiltonian.

In the most simple case for a ferromagnet, which we will study, the mechanics are goverened by the following Hamiltonian:

$$\mathcal{H} = -2J\sum \mathbf{s}_n\mathbf{s}_{n+1} - D\sum (s_n^z)^2. \tag{3.3}$$

We use this Hamiltonian to find a differential equation describing the motion, and plug in our spin wave ansatz. This will lead us to the dispersion relation, as a condition on our solution. Here the first term is the Heisenberg exchange term, as discussed in the theory section. The negative sign on the first term ensures that the prefered state of adjacent spins is in a parallel alignment, which is the case of ferromagnetism. Antiferromagnetism can be modeled simply by having the first term be positive. The second term is an anisotropy term, as discussed in the theory section. The effect of this term is to make the z axis the easy-axis, or the axis of preferred alignment. Thus, the magnetic moments will precess around the z axis. To understand this precession, we need to find the field that the individual lattice points are experiencing.

The Hamiltonian is related to the effective field experienced at a point on the lattice by the following relation;

$$\mathbf{H}_{eff} = -\nabla\mathcal{H}. \tag{3.4}$$

Thus at lattice point n, we have the following effective field :

$$\mathbf{H}_n^{eff} = 2J(\mathbf{s}_{n-1} + \mathbf{s}_{n+1}) + 2Ds_n^z\hat{z}. \tag{3.5}$$

A magnetic moment in an applied field will experience a net torque [10]. This leads us to a differential equation which governs the motion of the magnetic moment at any given lattice point n, found in [10]:

$$\frac{d\mathbf{s}_n}{dt} = \mathbf{s_n} \times \mathbf{H}_{eff}. \tag{3.6}$$

Combining Equation 3.6 with Equation 3.5, we obtain the following relation,

$$\begin{aligned}\frac{d\mathbf{s}_n}{dt} &= (s_n^y(2J(s_{n-1}^z + s_{n+1}^z) + s_n^y 2Ds_n^z - s_n^z 2J(s_{n-1}^y + s_{n+1}^y))\ \hat{x} \\ &+ (s_n^z 2J(s_{n-1}^x + s_{n+1}^x) - (s_n^x 2J(s_{n-1}^z + s_{n+1}^z) + s_n^x 2Ds_n^z))\ \hat{y} \\ &+ (s_n^x 2J(s_{n-1}^y + s_{n+1}^y) - s_n^y 2J(s_{n-1}^x + s_{n+1}^x))\ \hat{z}.\end{aligned} \tag{3.7}$$

Spin wave solutions to the lattice equation are solutions where the amplitude of each oscillation is the same. In particular, $s_i^z = s_j^z$ for all i, j in the lattice. For this reason we use the new coordinate, $s_n^+ = \frac{s_n^x + is_n^y}{|\mathbf{s}|}$. The introduction of this coordinate will permit us to focus in on the behavior of most interest, which is how the s_n^z coordinate changes, and how the "planar" component s_n^+ of the magnetic moment changes. Thus, our next step is to determine the rate of change of s_n^+, the declination of the n^{th} spin from the z axis. This is a quantity of interest because for the general spin wave solution, $\frac{ds_n^+}{dt} = \frac{ds_n^z}{dt} = 0$. However, if we introduce a perturbation, it should be possible for the declination of the spins to change over time. This behaviour would signal an ILM solution as opposed to the standard spin wave solution.

So to proceed, we find $\frac{ds_n^+}{dt}$. If we multiply both sides of Equation 3.7 by the imaginary unit i, after combining like terms, we can use Equation 2.8 to obtain the following relation:

$$\frac{ids_n^+}{dt} = s^+ 2J((s_{n-1}^z + s_{n+1}^z) - s_n^z 2J(s_{n-1}^+ + s_{n+1}^+) + 2Ds_n^z s^+. \tag{3.8}$$

Note that for the general spin wave solution, $\frac{ds_n^z}{dt} = 0$, because the spins remain at constant z amplitutude. With the differential equation for s_n^+, we can use the general spin wave ansatz discussed in Section 2.2 to find the dispersion relation for the general spin wave on a one-dimensional lattice with the Hamiltonian shown in Equation 3.3. We use as our ansatz the general spin wave solution given by the following equations; $s_n^+ = fe^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, $s_n^z = \sqrt{1 - f^2}$, where f is a constant, $\mathbf{k}$ is the wavenumber associated

with the spin wave solution, ω is the frequency associated with the spin wave solution, and $\mathbf{r}$ is a lattice node. Plugging these expressions in to Equation 3.8, and taking the lattice constant as equal to 1, we find in the one dimensional case that

$$\begin{aligned}\omega f e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} =& 2Jfe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2\sqrt{1-f^2} \\ &- \sqrt{1-f^2}2J(fe^{i(\mathbf{k}\cdot(\mathbf{r}-\hat{\mathbf{x}})-\omega t)} + fe^{i(\mathbf{k}\cdot(\mathbf{r}+\hat{\mathbf{x}})-\omega t)}) \\ &+ \sqrt{1-f^2}2Dfe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.\end{aligned} \tag{3.9}$$

By algebraic manipulation, we find that

$$\omega = 4J\sqrt{1-f^2} - 2J\sqrt{1-f^2}(e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}) + \sqrt{1-f^2}2D. \tag{3.10}$$

If we use Euler's formula to get $e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} = 2\cos(\mathbf{k}\cdot\hat{\mathbf{x}})$, we find that this relation can be written as

$$\frac{\omega}{2\sqrt{1-f^2}} = 2J - 2J(\cos(\mathbf{k}\cdot\hat{\mathbf{x}})) + D. \tag{3.11}$$

This is the one dimensional dispersion relation for a lattice where the lattice constant is equal to one unit.

This dispersion relation describes the spin wave on our one-dimensional lattice. It will also be an important tool in determining the stability of the spin wave solution. Note that if D, the strength of the anisotropy magnetic field, is equal to 0, then when $k = 0$, $\omega = 0$. Theoretically, this "gap" caused by $D > 1$ is necessary for the existence of ILMs. Furthermore, ILMs require nonlinear response, but this is taken care of, because our dispersion relation is cosinusoidal.

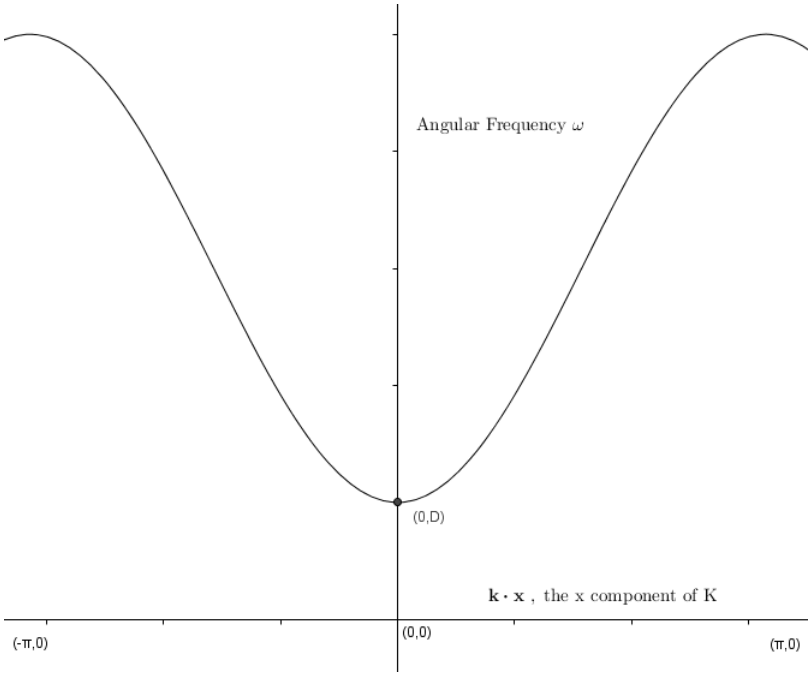


Figure 3.1: Angular frequency ω vs. wave number k. This is the dispersion relation of the 1-D spin wave given our Hamiltonian.

Chapter 4

Instability Derivation

We now analyze the stability of the spin wave given by $s_n^+ = fe^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, s_n^z = \sqrt{1-f^2}$ on the Bravais lattice with Hamiltonian in Section 2.3. We perform this analysis by introducing an additional function with a small local amplitude. The function we introduce is a small amplitude oscillation on the lattice. When we observe the mechanics of the spin wave-and-small perturbation combined state, we will determine under what conditions the amplitude of the perturbation decreases or increases. For instance, an exponential increase in the amplitude of the perturbation function would point to an unstable spin wave system, wherein small variations between lattice points results in the destruction of the spin wave state (possibly to be replaced by an ILMs solution). Alternatively, if the amplitude of the perturbation decreases exponentially with time, the spin wave system under consideration would be stable, with small perturbations dying out and the spin wave state remaining.

The perturbation function we aim to introduce is a function of the form

$$C_n = b_n + i\psi_n, \text{ such that}$$
$$b_n = be^{i(\kappa\cdot\mathbf{n}-mt)} \text{ and } \psi_n = \psi e^{i(\kappa\cdot\mathbf{n}-mt)}.$$

In this equation, $\mathbf{n}$ is the position on the lattice, b and ψ are constants such that $b, \psi << f$, where f is again the z-component of the magnetic moment. This is the equation of a wave with complex amplitude, with κ as the wave vector and m as the angular frequency.

We will plug in the new ansatz (spin wave plus perturbation) into the differential equation which defines the mechanics of this system, Equation 3.8. After algebraic manipulation, this equation will have two parts: real and imaginary. We separate into real and imaginary components, which allows us to form a system of dependent differential equations. This in turn permits us to solve for m^2, the square of the angular frequency of the perturbation. We permit m to take on values from the complex numbers. If $m = im_1$ is complex, the equation of the perturbation is

$$(b_n + i\psi_n)e^{i(\kappa\cdot\mathbf{x} - im_1 t)} = (b_n + i\psi_n)e^{i(\kappa\cdot\mathbf{x})}e^{mt}. \tag{4.1}$$

We notice that a term e^{mt} becomes a factor of the amplitude. In case $m_1 < -1$, this can result in a growing perturbation, and the eventual disintegration of the pure spin wave solution, which will be replaced by an ILM. Thus the main thrust of the following analysis is to determine when m, the angular momentum of the perturbation, is imaginary, because this signifies an instability of the spin wave solution to a perturbation. Practically speaking, this involves finding an equation giving m^2 in terms of system constants and wavevectors, and determining where this function is negative.

To begin, we form our new ansatz (note that s_n^z depends only on the real component of the perturbation):

$$s_n^+ = (f + b_n + i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}; s_n^z = \sqrt{1 - (f + b_n)^2} \tag{4.2}$$

and plug this into our original equation of motion, Equation 3.8, to get

$$\begin{aligned}\frac{id[(f+b_n+i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]}{dt} &= (f+b_n+i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2J(\sqrt{1-(f+b_{n-1})^2}+\sqrt{1-(f+b_{n+1})^2}\ ) \\ &- \sqrt{1-(f+b_n)^2}2J((f+b_{n-1}+i\psi_{n-1})e^{i(\mathbf{k}\cdot(\mathbf{r}-\hat{\mathbf{x}})-\omega t)}) \\ &- \sqrt{1-(f+b_n)^2}2J(f+b_{n+1}+i\psi_{n+1})e^{i(\mathbf{k}\cdot(\mathbf{r}+\hat{\mathbf{x}})-\omega t)}) \\ &+ (f+b_n+i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2D\sqrt{1-(f+b_n)^2}.\end{aligned} \tag{4.3}$$

We then expand the left side of that equation;

$$\begin{aligned}\frac{id[(f+b_n+i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]}{dt} &= (b_n'+i\psi_n')e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+(f+b_n+i\psi_n)(-i\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}) \\ &= (\omega f+ib_n'+\omega b_n-\psi_n'+i\psi_n\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}).\end{aligned} \tag{4.4}$$

To simplify and treat the right side of the equation, we must begin by making some approximations. First, since $b_n, \psi_n << f$, we feel justified in setting a second-order expression of these perturbation coefficients to 0. For instance, the expressions b_n^2 or $b_n\psi_n$ will be set to 0. Second, if we perform a Taylor expansion on the term $\sqrt{1-(f+b_n)^2}$, we find $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2} - \frac{b_n f}{\sqrt{1-f^2}}\ldots$. We will truncate this Taylor series at either one or two terms and proceed with the stability analysis using both cases. We will see a that a fundamental difference in behavior results from using $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2} - \frac{b_n f}{\sqrt{1-f^2}}$ as opposed to using $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2}$.

Taking the first of these two approximations, we rewrite the right-hand side of

Equation 4.3, getting:

$$\begin{aligned}(\omega f + ib_n' + \omega b_n - \psi_n' + i\psi_n\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &= (f + b_n + i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2J(\sqrt{1-(f+b_{n-1})^2}) \\ &+ (f + b_n + i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2J(\sqrt{1-(f+b_{n+1})^2}) \\ &- \sqrt{1-(f+b_n)^2}2J((f + b_{n-1} + i\psi_{n-1})e^{i(\mathbf{k}\cdot(\mathbf{r}-\hat{\mathbf{x}})-\omega t)}) \\ &- \sqrt{1-(f+b_n)^2}2J(f + b_{n+1} + i\psi_{n+1})e^{i(\mathbf{k}\cdot(\mathbf{r}+\hat{\mathbf{x}})-\omega t)}) \\ &+ (f + b_n + i\psi_n)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}2D\sqrt{1-(f+b_n)^2}.\end{aligned} \tag{4.5}$$

To solve this equation we must make liberal use of our dispersion relation, Equation 3.11. In particular, we use the dispersion relation to form the following three expressions:

$$\begin{aligned}\omega f &= 4Jf\sqrt{1-f^2} - 4Jf\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) + f\sqrt{1-f^2}2D \\ \omega b_n &= 4Jb_n\sqrt{1-f^2} - 4Jb_n\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) + b_n\sqrt{1-f^2}2D \\ i\psi_n\omega &= i4J\psi_n\sqrt{1-f^2} - i4J\psi_n\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) + i\psi_n\sqrt{1-f^2}2D\end{aligned} \tag{4.6}$$

If we divide both sides by $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, substitute the above expressions and subtract from the left side to the right, we are left with

$$
\begin{aligned}
ib_n' - \psi_n' = -2J\Bigg(& \frac{f^2 b_{n-1}}{\sqrt{1-f^2}} + \frac{f^2 b_{n+1}}{\sqrt{1-f^2}} + \frac{b_n b_{n-1} f}{\sqrt{1-f^2}} \\
& + \frac{b_n b_{n+1} f}{\sqrt{1-f^2}} + \frac{i\psi_n b_{n-1} f}{\sqrt{1-f^2}} + \frac{i\psi_n b_{n+1} f}{\sqrt{1-f^2}} \Bigg) \\
-2J\Bigg(& \sqrt{1-f^2} b_{n-1} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2} b_{n+1} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2} i\psi_{n-1} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} \\
& + \sqrt{1-f^2} i\psi_{n+1} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f^2}{\sqrt{1-f^2}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f^2}{\sqrt{1-f^2}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} \\
& - \frac{b_n f b_{n-1}}{\sqrt{1-f^2}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f b_{n+1}}{\sqrt{1-f^2}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} \\
& - \frac{b_n f i\psi_{n-1}}{\sqrt{1-f^2}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f i\psi_{n+1}}{\sqrt{1-f^2}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} \Bigg) \\
& - 2D\left(\frac{b_n f^2}{\sqrt{1-f^2}} + \frac{b_n^2 f}{\sqrt{1-f^2}} + \frac{i\psi_n b_n f}{\sqrt{1-f^2}} \right) \\
& + 4Jb_n\sqrt{1-f^2}\cos\left(\mathbf{k}\cdot\hat{\mathbf{x}}\right) + 4Ji\psi_n\sqrt{1-f^2}\cos\left(\mathbf{k}\cdot\hat{\mathbf{x}}\right).
\end{aligned}
\tag{4.7}
$$

Setting terms quadratic in perturbation coefficients to 0, we obtain

$$
\begin{aligned}
ib_n' - \psi_n' = -2J\Bigg(& \frac{f^2 b_{n-1}}{\sqrt{1-f^2}} + \frac{f^2 b_{n+1}}{\sqrt{1-f^2}} \Bigg) \\
-2J\Bigg(& \sqrt{1-f^2} b_{n-1} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2} b_{n+1} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2} i\psi_{n-1} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} \\
& + \sqrt{1-f^2} i\psi_{n+1} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f^2}{\sqrt{1-f^2}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{b_n f^2}{\sqrt{1-f^2}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} \Bigg) \\
& - 2D\left(\frac{b_n f^2}{\sqrt{1-f^2}} \right) \\
& + 4Jb_n\sqrt{1-f^2}\cos\left(\mathbf{k}\cdot\hat{\mathbf{x}}\right) + 4Ji\psi_n\sqrt{1-f^2}\cos\left(\mathbf{k}\cdot\hat{\mathbf{x}}\right).
\end{aligned}
\tag{4.8}
$$

We are now in a position where we can form two linear dependent differential equations. We do this by separating out the imaginary and the real components of the equations above. Applying Euler's formula to two terms in the equation and separating out the real and imaginary components, we get the following two equations:

$$\begin{aligned}-\psi_n' = &-2J\left(\frac{f^2 b_{n-1}}{\sqrt{1-f^2}} + \frac{f^2 b_{n+1}}{\sqrt{1-f^2}}\right) \\ &- 2J\left(\sqrt{1-f^2}b_{n-1}e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2}b_{n+1}e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} - \frac{2b_n f^2\cos(\mathbf{k}\cdot\hat{\mathbf{x}})}{\sqrt{1-f^2}}\right) \\ &- 2D\left(\frac{2b_n f^2}{\sqrt{1-f^2}}\right) + 4b_n J\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}})\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}b_n' = &-2J(\sqrt{1-f^2}\psi_{n-1}e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + \sqrt{1-f^2}\psi_{n+1}e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}) \\ &+ 4J\psi_n\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}).\end{aligned} \tag{4.10}$$

At this point, we wish to remove dependence on lattice points. Recall from earlier that $b_n = be^{i(\kappa\cdot\mathbf{n}-\omega t)}$ and $\psi_n = \psi e^{i(\kappa\cdot\mathbf{n}-mt)}$. Using this formula, dividing out by $e^{i(\kappa\cdot\mathbf{n}-mt)}$, and using Euler's formula we get the following two equations :

$$\begin{aligned}im\psi = &-2J\left(\frac{2f^2 b\cos(\kappa\cdot\hat{\mathbf{x}})}{\sqrt{1-f^2}}\right) \\ &- 2J\left(2\sqrt{1-f^2}b\cos(\kappa\cdot\hat{\mathbf{x}}+\mathbf{k}\cdot\hat{\mathbf{x}}) - \frac{2bf^2\cos(\mathbf{k}\cdot\hat{\mathbf{x}})}{\sqrt{1-f^2}}\right) \\ &- 2D\left(\frac{2bf^2}{\sqrt{1-f^2}}\right) + 4bJ\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}})\end{aligned} \tag{4.11}$$

and

$$\begin{aligned} -imb = -2J(2\sqrt{1-f^2}\psi\cos(\kappa\cdot\hat{\mathbf{x}}+\mathbf{k}\cdot\hat{\mathbf{x}}) \\ +4J\psi\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}). \end{aligned} \tag{4.12}$$

We can write these equations in matrix form as follows :

$$\begin{bmatrix} im & -2J(2\sqrt{1-f^2}\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}})) \\ & +4J\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) \\ 2J\left(\frac{2f^2(\cos(\mathbf{k}\cdot\hat{\mathbf{x}})-\cos(\kappa\cdot\hat{\mathbf{x}}))}{\sqrt{1-f^2}}\right) & \\ -2J\left(2\sqrt{1-f^2}\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}})\right) & \\ -2D\left(\frac{2f^2}{\sqrt{1-f^2}}\right)+4J\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) & -im \end{bmatrix}\begin{pmatrix} b \\ \psi \end{pmatrix} = 0$$

At this moment, we have a number of different parameters, and we recall their use briefly. The constants J and D refer to the strengths of forces in the Hamiltonian, and f is a constant which determines the planar projection of the spin wave. The variables k and ω describe the spin wave solution. The variables κ and m describe the perturbation. Since we have factored out dependence on ω, we now look for an equation which gives m^2 as a function of k and κ as variables, and may contain the constants f, J and D. We then wish to determine under what circumstances on k, κ, f, J and D we can force m to be imaginary. In linear algebra terms, we wish to find vectors $\binom{b}{\psi}$ such that we get nontrivial solutions to the above system of equations. To guarantee this, we require the determinant to be zero. So to proceed, we take the determinant of the above matrix, obtaining

$$
\begin{aligned}
m^2 = {} & 16J^2(1-f^2)(\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) - \cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}))^2 \\
& + 16J^2f^2(\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) - \cos(\kappa\cdot\hat{\mathbf{x}}))\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) \\
& - 16J^2f^2(\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) - \cos(\kappa\cdot\hat{\mathbf{x}}))\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) \\
& + 16DJf^2\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) - 8DJ\cos(\mathbf{k}\cdot\hat{\mathbf{x}}). \qquad (4.13)
\end{aligned}
$$

This is known as the stability equation of the system, because when m^2 is negative, we have that m is imaginary, so the perturbation will change in time. To complete our stability analysis, we also set up the stability matrix and corresponding stability equation when we take the approximation $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2}$. The stability matrix is as follows :

$$
\begin{bmatrix}
im & 2J\sqrt{1-f^2}\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) \\
 & -4J\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) \\
2J\sqrt{1-f^2}\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) & \\
-4J\sqrt{1-f^2}\cos(\mathbf{k}\cdot\hat{\mathbf{x}}) & -im
\end{bmatrix}
\begin{pmatrix} b \\ \psi \end{pmatrix} = 0
$$

and the corresponding stability equation is

$$
\frac{m^2}{4J^2(1-f^2)} = 2\cos(2(\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) - 4\cos(\mathbf{k}\cdot\hat{\mathbf{x}})\cos((\kappa+\mathbf{k})\cdot\hat{\mathbf{x}}) + 4\cos^2(\mathbf{k}\cdot\hat{\mathbf{x}}) + 2. \qquad (4.14)
$$

Chapter 5

Analysis

Thus far we have been careful about leaving in expressions such as $\mathbf{k} \cdot \hat{\mathbf{x}}$ and $\kappa \cdot \hat{\mathbf{x}}$, because if this research is extended to a two dimensional lattice, this will prove important. However, in the following section we adopt the notation $\mathbf{k} \cdot \hat{\mathbf{x}} = \mathbf{k}_x$ and $\kappa \cdot \hat{\mathbf{x}} = \kappa_x$ for simplicity's sake.

Recall that during the stability analysis, our ansatz for s_n^+ was $s_n^+ = \sqrt{1-(f+b_n)^2}$. To proceed with our analysis, we found the Taylor series corresponding to this function. We first performed the stability analysis using only the first term of the Taylor approximation. We first discuss the stability results obtained by using the approximation $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2}$.

As seen in the previous section, the stability analysis of the system resulted in the stability equation

$$\frac{m^2}{4J^2(1-f^2)} = 2\cos\left(2\kappa_{\mathbf{x}} + 2\mathbf{k}_{\mathbf{x}}\right) - 8\cos\left(\mathbf{k}_{\mathbf{x}}\right)\cos\left(\kappa_{\mathbf{x}} + \mathbf{k}_{\mathbf{x}}\right) + 4\cos^2\left(\mathbf{k}_{\mathbf{x}}\right) + 2. \quad (5.1)$$

If we wish to find where m^2 is negative, we take the derivative with respect to k and κ and find the critical points by setting these equations equal to 0. This function has critical points at $\sin\left(\kappa_{\mathbf{x}}\right) = 0$ and $\kappa_{\mathbf{x}} + \mathbf{k}_{\mathbf{x}} = 0$ modulo 2π. An analysis of these

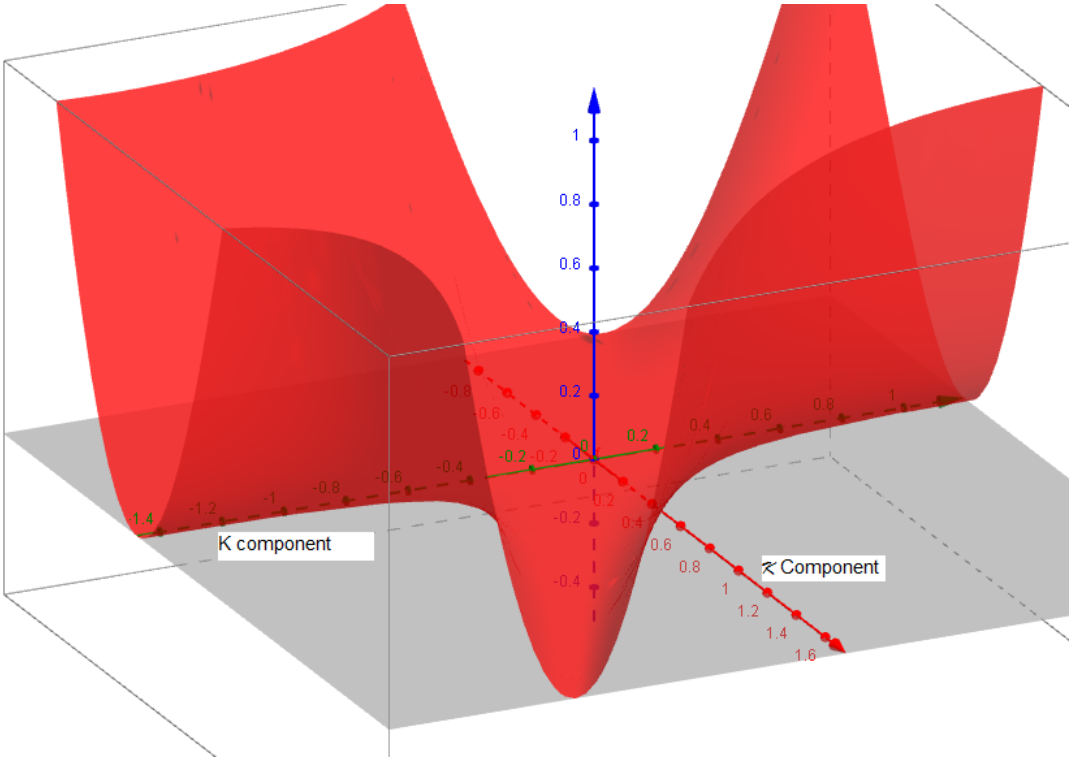


Figure 5.1: m^2 vs x component of k and x component of κ for the uniform mode. Here we see the minimum value is 0 at $(0, 0)$.

points show that they are never negative. In particular, the corresponding stability equation for the uniform mode $k = 0$ is

$$\frac{m^2}{4J^2(1-f^2)} = 2\cos(2\kappa_{\mathbf{x}}) - 8\cos(\kappa_{\mathbf{x}}) + 4\cos^2(\kappa_{\mathbf{x}}) + 2. \tag{5.2}$$

This function is never negative, and is 0 at $\kappa_x = 0$, which implies that the uniform mode is stable. This result was surprising, because concurrent research using numerical simulation of spin wave stability given by English and Kimock [11] suggests that spin waves are unstable. While the motivation for the uniform mode being unstable often derives from terms not included in our Hamiltonian, such as a demagnetizing field, the uniform mode nevertheless is an easy case to examine, and so we refine our approximations to attempt to find instability, if possible, in the uniform mode. Thus, the rest of this analysis will deal only with the uniform mode.

This prompts us to revise our assumptions, and in particular to use a more exact approximation. If we instead make the approximation $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2} - \frac{fb_n}{\sqrt{1-f^2}}$, we obtain Equation 4.13 as our stability equation, and different behaviour emerges. While Equation 4.13 is difficult and unweildy to analyze in full generality, we can still determine that instability is present in the system. For instance, if we take the stability equation for the uniform mode $k = 0$, by checking at different values for J, D and f, we can certainly find cases where $m^2 < 0$, as we see below in Figure 5.2.

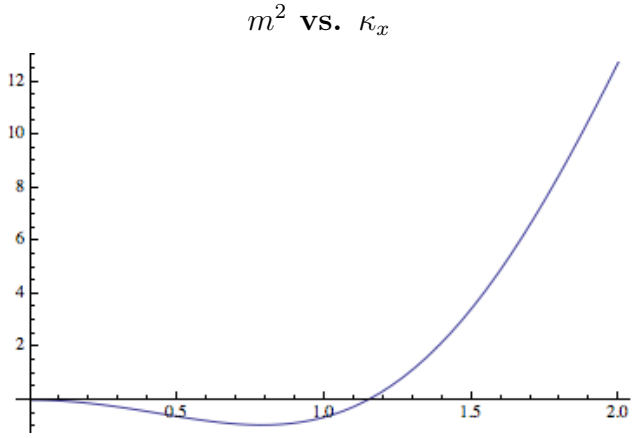


Figure 5.2: m^2 vs. κ_x, where $J = 1$, $f = .60$, $D = 2$, for the uniform mode $k = 0$.

We use Mathematica to plot m^2 as a function of the $\hat{\mathbf{x}}$ component of k and κ for different values of f and D, with $J = 1$. Figure 5.2 clearly demonstrates that the system is unstable with respect to our perturbation function for certain values of κ. However, past $\kappa_x = 1.2$, m^2 is positive. This demonstrates that our system is unstable with respect to perturbations with small κ, but for $\kappa > 1.2$, the system is stable. In our analysis of these functions, there are two values of interest: the minimum value of the stability equation and the x-intercept of the stability equation.

The minimum value of the stability equation, if negative, measures how quickly the system changes. A system with a lower minimum could transform from spin wave behaviour to ILM behaviour much more quickly than a system with a minimum close

to 0. In some sense, the minimum value measures how quickly the spin wave state responds to the perturbation.

The x-intercept of the stability function (discounting the trivial case of $(0, 0)$) expresses the wavevectors κ such that the spin wave is instable with respect to the perturbation. Thus, as the x-intercept increases, the system is unstable with respect to more and more perturbations. For the following analysis, we observe the uniform mode $k = 0$ and set the Heisenberg constant $J = 1$. While setting $k = 0$ allows us to focus on the uniform mode, which is a particular physical spin wave state, setting the exchange constant $J = 1$ is simply for easier analysis.

As D is held fixed and f is increased towards 1, we note that the minimums of the functions decrease, as seen in figures 5.3 and 5.4.

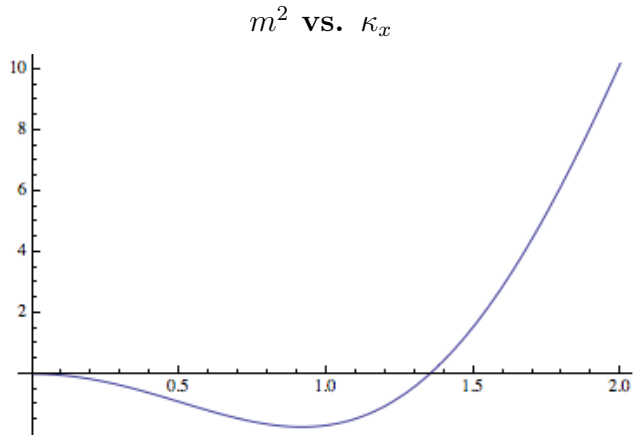


Figure 5.3: m^2 vs. κ_x, where $J = 1$, $f = .65$, $D = 2$, for the uniform mode $k = 0$.

To investigate this more thoroughly, in Figure 5.5 we plot the minimum value attained by m^2 over different values. The three curves correspond to $D = .5$, , $D = 1$ and $D = 2$, while f increases from 0 to 1. Figure 5.5 clearly shows that as D is held constant, an increase in f leads to lower minimums, corresponding to greater sensitivity to perturbations. Note that all three curves corresponding to different values of D are cut off below a certain value. From this we see that when f is sufficiently low, the spin wave solution is stable. Recall that f is the planar

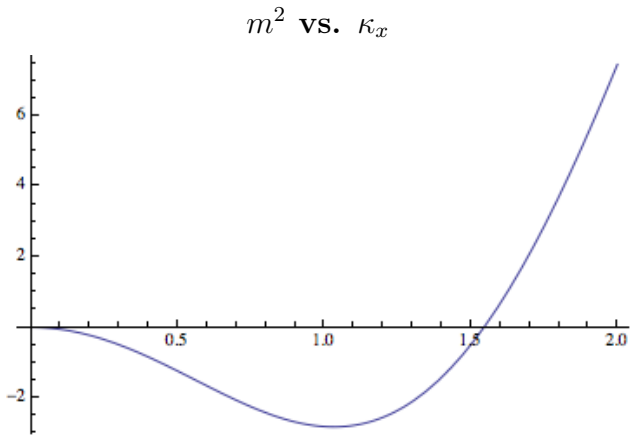


Figure 5.4: m^2 vs. κ_x, where $J = 1$, $f = .70$, $D = 2$, for the uniform mode $k = 0$.

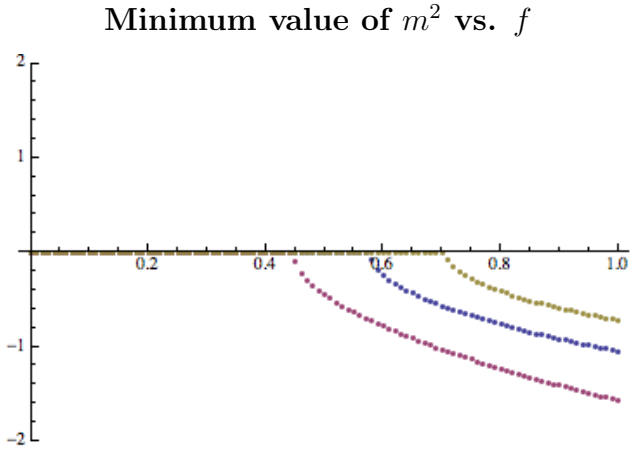


Figure 5.5: Minimum values as f increases from 0 to 1. The red line is for $D = .5$, blue is for $D = 1$, and green is for $D = 2$, for the uniform mode $k = 0$.

component of the spin wave, so a low f means spins mostly aligned with the z-axis. Furthermore, it seems to be the case that as D is increased from .5 to 1 and on to 2, the region of instability shrinks. Thus, the lattice is unstable with respect to more perturbations when D is small.

If we instead examine the x-intercept, we find a similar dependence on f and D. If we plot x-intercept as a function of f, holding D constant, we obtain the graph shown in Figure 5.6.

Different values of f correspond to different points on the graph, as D is held constant.

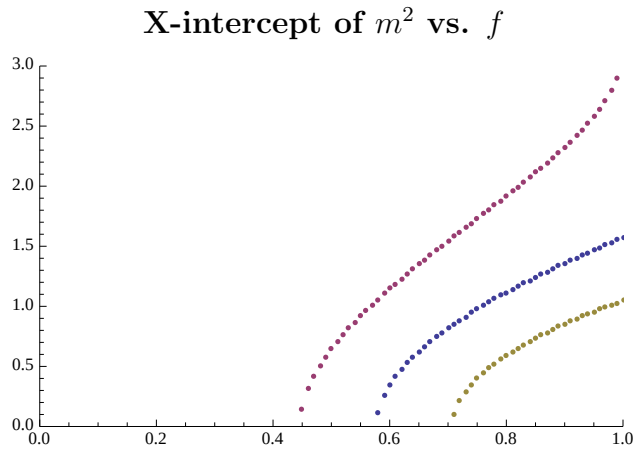


Figure 5.6: X-intercept as f increases from 0 to 1. The red line is for $D = .5$, blue is for $D = 1$, and green is for $D = 2$, for the uniform mode $k = 0$.

Here we see that as f is increased, the x-intercept also increases. Since a greater x-intercept corresponds to a larger portion of the m^2 curve being negative, a greater x-intercept implies that the system is unstable with respect to a larger range of perturbations. Note that D also has an effect. If we look at constant f, we note that as D increases, the x-intercept decreases.

From Figures 5.5 and 5.6 we gain some information as to the stability of our system. For the values sampled, as f increases towards 1, the system responds more and more to instability. This is shown in the first graph, as the minimum value decreases as f increases. The first graph also demonstrates that as D increases, stability also increases, leading to the offset between the lines of $D = .5$, $D = 1$ and $D = 2$. In the second figure we see that the regions of instability (that is, the number of perturbations that the system is unstable with respect to) increases as f increases. As D increases, the system becomes stable with respect to more and more perturbations, and responds more slowly.

These results fit well with our intuition. Since D represents the strength of the anisotropy field pointing in the easy-axis direction, a very strong magnetic field should be difficult to perturb. This is precisely what is shown (though over a very small

sampling of points) in the above results. However, we note that as D decreases, not only do perturbations disturb the spin wave state faster, there are more perturbations which cause the system to be unstable. Furthermore, recall that f is the planar component of the spin at any given lattice point in the spin wave. If $f = 0$, we are in the ground state of the system, which must be stable. As more and more energy is added, f increases, which represents the spins declining further and further from the eazy-axis into the plane. It makes sense for a state with higher energy to be closer to an unstable state. These results are to some extent corroborated by concurrent simulations run by Kimock and English [11], which show the same dependence on f.

Chapter 6

Conclusion

This thesis has attempted to analytically determine the conditions under which spin wave solutions on 1-dimensional ferromagnets are unstable. To perform this analysis, first we determined the dispersion relation of the spin wave solution on a 1-dimensional lattice when working with a Hamiltonian containing the Heisenberg exchange term and an anisotropy term.

After determining the dispersion relation for the spin wave, we added a perturbation function of the form $(b + i\psi)e^{i(\kappa \cdot \mathbf{x} - mt)}$ onto the amplitude of the S^+ components of the magnetic moments at every lattice point. The main thrust of this analysis was to determine when m, the angular frequency of the perturbation, was imaginary, as this leads to an exponential increase in the amplitude of the perturbation function. A time-dependent increase in the amplitude of the perturbation function corresponds to motion is intrinsically different from the spin wave solution, and so represents an instability of the spin wave solution.

Not only have we determined that spin waves on a 1-dimensional lattice are unstable under certain parameters, but we have also determined several dead-ends, where stability is predicted by the equations, in contradiction to simulation. In particular, the results section shows clearly that for instability to be present in

the spin wave solution, we are required in our analysis to make the approximation $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2} - \frac{fb_n}{\sqrt{1-f^2}}$; if we truncate this Taylor expansion at one term, the analysis tells us that the spin wave solution is stable. This runs counter to many numerical simulations, found in concurrent work by Kimock and English [11], as well as [12], [13] and [14]. This permits us to conclude that if additional research is to be performed, researchers should use our approximation of $\sqrt{1-(f+b_n)^2} = \sqrt{1-f^2} - \frac{fb_n}{\sqrt{1-f^2}}$ or better.

Chapter 7

Suggestions for Further Research

Though we have analytically demonstrated instability in this system, in a certain sense the work has just begun. There are several accessible and natural extensions on to this thesis.

First and foremost, there remains much to be done on the analysis of the stability equation. At this point, though we have investigated the behaviour for several values of J, D, f and κ, we do not have a fully satisfactory answer for how the x-intercept of this function depends on these constants. The brief analysis in this thesis was only done on the uniform mode $\mathbf{k} = 0$. Thus we have little knowledge on how $\mathbf{k}$ affects the system, and how $\mathbf{k}$ and κ could interact and lead to instability. Another task which remains undone is to categorize and understand the behaviour of the system when paired with a perturbation under which it is unstable. That is, this thesis has begun pointing out which perturbations result in a loss of spin wave solutions. However, we have not described what actually occurs in those situations; does the system move randomly or chaotically? Can the system return to a spin wave state once it has been perturbed by a significant perturbation? What parameters can affect the rapidity and speed at which the spin wave solution breaks down? All of these questions go deeper into the actual nature of the spin wave instability, more than just determining

at what parameters the spin wave solution is unstable.

Secondly, in order to link these theoretical results more closely with experiment and numerical simulation, another important direction for further research to take is to expand the theoretical analysis presented here into 2-dimensional ferromagnetic lattices. Though it is not anticipated that this will drastically alter the stability of the system, additional terms will appear in the stability equation, and as this has not yet been thoroughly treated, the step to a 2-dimensional lattice is not trivial. Furthermore, the entirety of this thesis was done under the assumption that our Bravais lattice was infinite in length. To better reflect experiment, it may prove useful to consider 2-dimensional finite lattices.

If altering the Bravais lattice under consideration is a second path for further research, a third path would be to add in magnetic fields that more accurately represent the mechanics of a physical magnet. In particular, many models of ferromagnets such as in English [5] make use of a third term in the Hamiltonian, which represents the demagnetization field due to the ferromagnet. This term is essential for fully describing the magnetic field at a point in a ferromagnetic crystal. The addition of this term would be a great step forward, and if both a demagnetization term and a finite 2-dimensional lattice are accounted for, this theoretical analysis can form the basis for experiments on crystal lattices.

Lastly, it has been postulated that there are other stable, non-spin wave solutions to the equations of motion of a ferromagnet called skyrmions. Furthermore, skyrmions have been theoretically shown to be distinct from spin waves insofar as one cannot turn into the other without overcoming a significant energy barrier. This entire analysis on the stability of spin waves could be repeated for skyrmion solutions instead.

Bibliography

[1] Felix Bloch. *Z. Physik*, 206, 1930.

[2] F.J. Dyson. *Selected Papers of Freeman Dyson with Commentary*. American Mathematical Society. American Mathematical Society, 1996.

[3] R. Pauthenet. Experimental verification of spinwave theory in high fields (invited). *Journal of Applied Physics*, 53(11), 1982.

[4] M. Plihal, D. L. Mills, and J. Kirschner. Spin Wave Signature in the Spin Polarized Electron Energy Loss Spectrum of Ultrathin Fe Films: Theory and Experiment. *Physical Review Letters*, 82:2579–2582, March 1999.

[5] Lars English. *Studies of Intrinsic Localized Spin-Wave Modes in Antiferromagnetic Crystals*. PhD thesis, Cornell University, 2003.

[6] R. Lai and A. J. Sievers. Modulational instability of nonlinear spin waves in easy-axis antiferromagnetic chains. *Phys. Rev. B*, 57:3433–3443, Feb 1998.

[7] N.W. Ashcroft and N.D. Mermin. *Solid State Physics*. HRW international editions. Holt, Rinehart and Winston, 1976.

[8] D.J. Griffiths. *Introduction to Quantum Mechanics*. Pearson international edition. Pearson Prentice Hall, 2005.

[9] S.F. Tuan and J.J. Sakurai. *Modern quantum mechanics*. Pearson, 2005.

[10] D.J. Griffiths. *Introduction to Electrodynamics*. Prentice Hall, 1999.

[11] Lars English and Ben Kimock. Private correspondence.

[12] S. Rakhmanova and D. L. Mills. Nonlinear spin excitations in finite heisenberg chains. *Phys. Rev. B*, 54:9225–9231, Oct 1996.

[13] Svetlana V. Rakhmanova and Andrei V. Shchegrov. Intrinsic localized modes of bright and dark types in ferromagnetic heisenberg chains. *Phys. Rev. B*, 57:R14012–R14015, Jun 1998.

[14] S. Rakhmanova and D. L. Mills. Intrinsic localized spin waves in classical one-dimensional spin systems: Studies of their interactions. *Phys. Rev. B*, 58:11458–11464, Nov 1998.